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Goodness-of-fit tests for the pseudo-Poisson Distribution

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ABSTRACT
Bivariate count models having one marginal and the other conditionals being of the Poissons form are called pseudo-Poisson distributions. Such models have simple, flexible dependence structures, possess fast computation algorithms, and generate a sufficiently large number of parametric families. It has been strongly argued that the pseudo-Poisson model will be the first choice to consider in modeling bivariate over-dispersed data with positive correlation and having one of the marginal equi-dispersed. Yet, before we start fitting, it is necessary to test whether the given data is compatible with the assumed pseudo-Poisson model. Hence, we derive and propose a few Goodness-of-Fit tests for the bivariate pseudo-Poisson distribution in the present note. Also, we emphasize two tests, a lesser-known test based on the Supremes of the absolute difference between the estimated probability generating function and its empirical counterpart. A new test has been proposed based on the difference between the estimated bivariate Fisher dispersion index and its empirical indices. However, we also consider the potential of applying the bivariate tests that depend on the generating function (like the Kocherlakota and Kocherlakota(K&K) and Muñoz and Gamero (M&G) tests) and the univariate Goodness-of-Fit tests (like the Chi-square test) to the pseudo-Poisson data. However, we analyze finite, large, and asymptotic properties for each of the tests considered. Nevertheless, we compare the power (bivariate classical Poisson and Conway-Maxwell bivariate Poisson as alternatives) of each of the tests suggested and also include examples of application to real-life data. In a nutshell, we are developing an R package that includes a test for the compatibility of the data with the bivariate pseudo-Poisson model.

KEYWORDS
Goodness-of-Fit test; Bivariate pseudo-Poisson; Marginal and Conditional distributions; Neyman Type A distribution; Thomas distribution

1. Introduction

Indeed, Goodness-of-Fit (GoF) test is a statistical procedure to test whether the given data is compatible with the assumed distribution. Any GoF test requires the following three Steps: (1) Identifying the unique characteristic of the assumed model (Examples: Distribution function, generating function, or density function); (2) Compute the empirical version of the assumed characteristic; (3) With the pre-assumed measure(Examples: L_1 - or L_2-space), measure the distance between assumed item in Step

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(1) and its empirical one, in Step (2). A rejection region can be computed with a given level, and the cut-off value for the distance measure is determined. However, if the rejection region can not be derived explicitly, one can use the Bootstrapping technique to generate a critical region. The general steps required to simulate a rejection region using Bootstrapping are discussed more in Section 4. We refer to Meintanis [16] and Nikitin [18] for a detailed discussion of the GoF tests, which involve the aforementioned steps. Besides, there do exist or can be constructed tests which are not based on a unique characteristic of the assumed distribution. For example, considering the univariate Poisson distribution, a GoF test exists that depends on the Fisher index of depression. We also know that the Poisson distribution belongs to the class of equi-dispersed models, but this property does not characterize the Poisson distribution. Hence, such tests, which are not based on a unique characteristics of the assumed distributions are not consistent tests.

The literature on GoF tests for bivariate count data is sparse. For the classical bivariate and multivariate Poisson distributions, a GoF test using the probability generating function is discussed by Muñoz and Gamero [20] and Muñoz and Gamero [21]. For a recent review of the available bivariate GoF tests and a new test using the differentiation of the probability generating function, see Muñoz [19].

In the following sections, we are starting with a test defined in Kocherlakota and Kocherlakota [12] and a few bivariate GoF tests reviewed in Muñoz [19]. In addition to the classical GoF tests using probability generating function (p.g.f.), we considered a less known test, which will be the Supremum of the absolute difference between estimated p.g.f. and empirical ones. In addition, we are introducing a non-consistent tests which are based on the moments, in particular, defining test taking difference of estimated bivariate Fisher index and its empirical counterpart. We examine each test's finite, large, and asymptotic properties and recommend a few tests based on their power and robustness analysis.

Before discussing GoF tests, we would like to make a few remarks on the bivariate pseudo-Poisson model and its relevance in the literature. Finally, we refer to Arnold and Manjunath [2] and Arnold et al.[3] for classical inferential aspects, characterization, Bayesian analysis, and also an example of applications of the bivariate pseudo-Poisson model.

2. Bivariate pseudo-Poisson models

In the following we will be discussing the bivariate pseudo-Poisson model, see Arnold and Manjunath [2] page 2307.

Definition 2.1. A 2-dimensional random variable (X, Y) is said to have a bivariate pseudo-Poisson distribution if there exists a positive constant λ_1 such that $X \sim \mathscr{P}(\lambda_1)$ and a function $\lambda_2 : \{0, 1, 2, ...\} \to (0, \infty)$ such that, for every non-negative integer x, $Y|X = x \sim \mathscr{P}(\lambda_2(x))$.

Here we restrict the form of the function $\lambda_2(x)$ to be a polynomial with unknown coefficients. In particularly the simple form we assume is that $\lambda_2(x) = \lambda_2 + \lambda_3 x$, then the above bivariate distribution will be of the form

$$X \sim \mathscr{P}(\lambda_1), \tag{1}$$

and

$$Y|X = x \sim \mathscr{P}(\lambda_2 + \lambda_3 x). \tag{2}$$

The parameter space for this model is $\{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 > 0, \lambda_2 > 0, \lambda_3 \geq 0\}$. The case in which the variables are independent corresponds to the choice $\lambda_3 = 0$. The probability generating function (p.g.f.) for this bivariate pseudo-Poisson distribution is given by

$$G(t_1, t_2) = e^{\lambda_2(t_2-1)} e^{\lambda_1[t_1 e^{\lambda_3(t_2-1)}-1]};\ t_1, t_2 \in \mathbb{R}. \tag{3}$$

Remark 1. As noted in Arnold and Manjunath [2], for the case $\lambda_2 = 0$, the bivariate pseudo-Poisson distribution reduces to the bivariate Poisson-Poisson distribution. The corresponding Poisson-Poisson distribution was initially introduced by Leiter and Hamdani [14] in modeling traffic accidents, and fatalities count data. The bivariate pseudo-Poisson model is a generalization of the Poisson-Poisson distribution.

The joint p.g.f. in equation (3) deduces to

$$G_{II}(t_1, t_2) = e^{\lambda_1[t_1 e^{\lambda_3(t_2-1)}-1]};\ t_1, t_2 \in \mathbb{R}. \tag{4}$$

Now, the marginal p.g.f. of Y is

$$G(1, t_2) = G_Y(t_2) = e^{\lambda_2(t_2-1)} e^{\lambda_1[e^{\lambda_3(t_2-1)}-1]};\ t_2 \in \mathbb{R}. \tag{5}$$

Note that, in general, the p.g.f. in equation (4) can not be simplified to compute all marginal probabilities. Yet, we can use equation (4) to derive a few marginal probabilities of Y. The derivation of marginal probability of Y is demonstrated for $Y = 0, 1, 2, 3$ in Appendix A, and one can still extend the mentioned procedure to get albeit complicated values for the probability that Y assumes any positive value. Besides, the derivation of the other conditional distribution of the bivariate pseudo-Poisson, i.e., $f(x|y)$, has been included in Appendices B.

In the following sections, we discuss a few one-dimensional distributions that are derived from the bivariate pseudo-Poisson for the case $\lambda_2 = 0$. Moreover, the derived univariate distributions have classical relevance to the two parameters, Neyman Type A and Thomas distribution.

2.1. Neyman Type A distribution

As noted in Arnold and Manjunath [2], in the case in which $\lambda_2 = 0$ the marginal distribution is a Neyman Type A distribution with λ_3 being the index of clumping (see page 403 of Johnson, Kemp, and Kotz [10]). It can also be recognized as a Poisson mixture of Poisson distributions. Now, the marginal mass function of Y is given by

$$P(Y = y) = \frac{e^{-\lambda_1} \lambda_3^y}{y!} \sum_{j=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_3})^j j^y}{j!};\ y = 0, 1, 2, ... \tag{6}$$

i.e., Y has a Poisson distribution with the parameter λ_1 while λ_1 is also a Poisson distribution with the parameter λ_3. We refer to Glesson and Douglas [7] and Johnson, Kemp and Kotz [10] Section 9.6 for applications and inferential aspects of the Neyman Type A distribution.

2.2. Thomas distribution

Consider the joint probability generating function defined in equation (4), i.e.,

$$G_{II}(t_1, t_2) = e^{\lambda_1[t_1 e^{\lambda_3(t_2-1)}-1]};\ t_1, t_2 \in \mathbb{R}. \tag{7}$$

Take $t_1 = t_2 := t$ and the above p.g.f. deduces to

$$G^*(t) = G(t, t) = e^{\lambda_1[te^{\lambda_3(t-1)}-1]};\ t \in \mathbb{R}. \tag{8}$$

Note that the above univariate p.g.f. is the p.g.f. of the Thomas distribution with parameter λ_1 and λ_3. The probability mass function of the Thomas distribution is given as

$$P(Z = z) = \frac{e^{-\lambda_1}}{z!}\sum_{j=1}^{z}\binom{z}{j}(\lambda_1 e^{-\lambda_3})^j (j\lambda_3)^{z-j},\ z = 0, 1, 2, ... \tag{9}$$

For further, applications and inferential aspects of the Thomas distribution, we refer to Glesson and Douglas [7] and Johnson, Kemp and Kotz [10]) in Section 9.10.

Remark 2. The Neyman Type A and the Thomas distribution have historical relevance in modeling plant and animal populations. For example, suppose that the number of clusters of eggs an insect lays and the number of eggs per cluster have specified probability distributions. Then, for the Neyman Type A distribution and Thomas distributions, the number of clusters of eggs laid by the insect follows a Poisson distribution with parameter λ_1. For the Neyman Type A, the number of eggs per cluster is also a Poisson distribution with parameter λ_3. But for the Thomas distribution, the parent of the cluster is always to be present with the number of eggs(offspring) and which has a shifted Poisson distribution with support $\{1, 2, 3, ...\}$ and the parameter λ_3. Note that Neyman Type A and Thomas distributions can be generated by a mixture of distributions and also a random sum of random variables.

Consider that the mixing distribution is a Poisson with parameter λ_1 with the mixture has a Poisson with parameter λ_3, then the resultant random variable has a Neyman Type A distribution. In the sequel, if the mixing distribution is a Poisson with parameter λ_1 and the jth distribution in the mixture has a distribution of the form $j + Y(j)$, where $Y(j)$ has a Poisson with parameter $j\lambda_3$ then the resultant random variable has a Thomas distribution.

However, for a random sum of random variables (also known as Stopped-Sum distributions), let us consider that the size N of the initial generation is a random variable and that each individual i of this generation independently gives a random variable Y_i, where $Y_1, Y_2, ...$ has a common distribution. Then the total number of individuals is $S_N = Y_1 + ... + Y_N$. For the case that N is a Poisson random variable with parameter λ_1 and Y_i is a Poisson random variable with parameter λ_3 then the random sum S_N

has a Neyman Type A distribution. However, if Y_i is a shifted Poisson with parameter λ_3 and support $\{1, 2, 3, ...\}$, then the random sum S_N has a Thomas distribution.

Remark 3. The other conditional mass function, i.e., the conditional mass function of X given $Y = y$, is recognized in Section 5 of Leiter and Hamdan [14] and in Appendix 3 of Arnold and Manjunath [2]. However, in Appendix B in the current note, we have derived the conditional mass function and identified the expression as a Stirling number of the second kind.

3. Goodness-of-Fit tests

In the following section, we discuss GoF tests, which are based on the moments (non-consistent tests), on unique characteristics (consistent tests), and a simple classical χ^2 Goodness-of-Fit test.

3.1. New test based on moments

In the following, we will be extending an univariate GoF test based on the Fisher index to the bivariate case. We know that for a multivariate distribution, the Fisher index of dispersion is not uniquely defined. However, in the following, we use the definition of the multivariate Fisher dispersion given by Kokonendji and Puig [13] in Section 3 as for any d-dimensional discrete random variable $\mathbf{Z}$ with mean vector $E(\mathbf{Z})$ and covariance matrix $Cov(\mathbf{Z})$ the generalized dispersion index is

$$GDI(\mathbf{Z}) = \frac{\sqrt{\mathbf{E(Z)}}^{\mathbf{T}}\mathbf{Cov(Z)}\sqrt{\mathbf{E(Z)}}}{\mathbf{E(Z)^T E(Z)}}. \tag{10}$$

For the bivariate pseudo-Poisson model, definite the random vector $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})^{\mathbf{T}}$ for and the moments are (c.f. Arnold and Manjunath [2] page 2309-2310)

$$E(\mathbf{Z}) = (\lambda_1, \lambda_2 + \lambda_3\lambda_1)^T, \tag{11}$$

$$cov(\mathbf{Z}) = \begin{bmatrix} \lambda_1 & \lambda_1\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2 + \lambda_3\lambda_1 + \lambda_3^2\lambda_1 \end{bmatrix}.$$

Now, using the definition given in Kokonendji and Puig [13] page 183, dispression index for the bivariate pseudo-Poisson is

$$\begin{aligned} GDI(\mathbf{Z}) &= \frac{\lambda_1^2 + 2\lambda_1^{\frac{3}{2}}\lambda_3\sqrt{\lambda_2 + \lambda_3\lambda_1} + (\lambda_2 + \lambda_3\lambda_1)(\lambda_2 + \lambda_3\lambda_1 + \lambda_3^2\lambda_1)}{\lambda_1^2 + (\lambda_2 + \lambda_3\lambda_1)^2} \\ &= 1 + \frac{2\lambda_1^{\frac{3}{2}}\lambda_3\sqrt{\lambda_2 + \lambda_3\lambda_1} + (\lambda_2 + \lambda_3\lambda_1)\lambda_3^2\lambda_1}{\lambda_1^2 + (\lambda_2 + \lambda_3\lambda_1)^2} > 1, \end{aligned} \tag{12}$$

which indicates over-dispersion.

For the corresponding sample version, consider the n sample observations $\mathbf{Z}_1 = (X_1, Y_1)^T$,..., $\mathbf{Z}_n = (X_n, Y_n)^T$ from the bivariate pseudo-Poisson distribution. Now, denote $\overline{\mathbf{Z}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{Z}_i = (\overline{X}, \overline{Y})^T$ and $\widehat{cov(\mathbf{Z})} = \frac{1}{n-1}\sum_{i=1}^{n} \mathbf{Z}_i \mathbf{Z}_i^T - \overline{\mathbf{Z}}_n \overline{\mathbf{Z}}_n^T$ are sample mean vector and sample covariance matrix, respectively. Then the empirical bivariate dispersion index is

$$\widehat{GDI(\mathbf{Z})}_n = \frac{\sqrt{\overline{\mathbf{Z}}_n^T}\,\widehat{cov(\mathbf{Z})}\sqrt{\overline{\mathbf{Z}}_n}}{\overline{\mathbf{Z}}_n^T \overline{\mathbf{Z}}_n}. \tag{13}$$

According to Theorem 1 in Kokonendji and Puig [13] page 184, as $n \to \infty$, $\sqrt{n}\{\widehat{GDI(\mathbf{Z})}_n - GDI(\mathbf{Z})\} \sim N(0, \sigma_g^2)$, where $\sigma_g^2 = \Delta^T \Gamma \Delta$;

$$\Gamma = \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0}, \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} var(X) & cov(X,Y) \\ cov(X,Y) & var(Y) \end{bmatrix}.$$

A new bivariate GoF test for the count data based on the Fisher dispersion index is

$$FI_n^{(.)} = \sqrt{n}\{\widehat{GDI(\mathbf{Z})}_n - GDI(\mathbf{Z})\}, \tag{14}$$

and the null hypothesis is rejected for large absolute values of $F_n^{(.)}$. The asymptotic distribution of the test statistic is

$$\frac{\widehat{GDI(\mathbf{Z})}_n - GDI(\mathbf{Z})}{\frac{\sigma_g}{\sqrt{n}}} \sim^{asy.} N(0,1), \text{ as } n \to \infty. \tag{15}$$

For the detailed proof, c.f. Theorem 1 in Kokonendji and Puig [13] page 184. However, for the two sub-models of the bivariate pseudo-Poisson model, i.e., when $\lambda_2 = \lambda_3$ is sub-model I and when $\lambda_2 = 0$ is sub-model II the new test statistics are

$$FI_n^{(SI)} = \sqrt{n}\{\widehat{GDI(\mathbf{Z})}_n - GDI^{(SI)}(\mathbf{Z})\}, \tag{16}$$

and

$$FI_n^{(SI)} = \sqrt{n}\{\widehat{GDI(\mathbf{Z})}_n - GDI^{(SII)}(\mathbf{Z})\}, \tag{17}$$

where

$$GDI^{(SI)}(\mathbf{Z}) = 1 + \frac{2\lambda_1^{\frac{3}{2}}\lambda_3^{\frac{3}{2}}\sqrt{1+\lambda_1} + (1+\lambda_1)\lambda_3^3\lambda_1}{\lambda_1^2 + \lambda_3^2(1+\lambda_1)^2}, \quad (18)$$

where

$$GDI^{(SI)}(\mathbf{Z}) = 1 + \frac{2\lambda_1^{\frac{3}{2}}\lambda_3^{\frac{3}{2}}\sqrt{\lambda_1} + \lambda_3^3\lambda_1^2}{\lambda_1^2 + \lambda_3^2\lambda_1^2}. \quad (19)$$

One can derive test statistic $FI_n^{(SI)}$ and $FI_n^{(SII)}$. The estimated dispersion index can be obtained by plugging in the m.l.e estimates of λ_i, $i = 1, 2, 3$. Also, due to the invariance and asymptotic properties of the m.l.e. estimates, the proposed test statistics are normally distributed (with appropriate scaling). For large sample sizes, the null hypothesis is rejected whenever the test statistic absolute value exceeds the standard normal quantile value. In Section 4, we analyze the finite, large, and asymptotic behavior of the proposed test statistic.

In addition, using bootstrapping techniques, one can simulate the distribution of the above test, and then testing for normality will also produce a robust GoF fit test.

3.2. Test based on the unique characteristic

In the following we consider a few test statistics for the full, sub-model I and sub-model II.

3.2.1. Muñoz and Gamero (M&G) method

The GoF tests for a bivariate random variable based on the finite sample size are limited. This is due to difficulty in deriving closed form expression for the critical region under finite sample size. Yet, in the following, we use the finite sample size test suggested in Muñoz and Gamero [20] for the classical bivariate Poisson distribution is used to construct the GoF test for the bivariate pseudo-Poisson distribution. For a finite sample test based on the p.g.f. to test GoF for the univariate Poisson, we refer to Rueda et al. [22]. Furthermore, using the bootstrapping technique, the critical region for the test is simulated and illustrated with an example in Section 4.

Let (X, Y) be a bivariate random variable with p.g.f. $G(t_1, t_2; \lambda_1, \lambda_2, \lambda_3)$, $(t_1, t_2)^T \in [0, 1]^2$. For the given data set (X_i, Y_i), $i = 1, ..., n$, we denote by $G_n(t_1, t_2) = \frac{1}{n}\sum_{i=1}^{n} t_1^{X_i} t_2^{Y_i}$ an empirical counterpart of the bivariate p.g.f.. According to Muñoz and Gamero [20] a reasonable test for testing the compatibility of the assumed density should reject the null hypothesis for large values of given statistic

$$T^{(.)}_{P,n,w}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = \int_0^1 \int_0^1 g_n^2(t_1, t_2; \hat{\lambda}_1, \hat{\lambda}_2 \hat{\lambda}_3) w(t_1, t_2) dt_1 dt_2, \quad (20)$$

where $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ are consistent estimators of λ's and $g_n(t_1, t_2; \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = \sqrt{n}\{G_n(t_1, t_2) - G(t_1, t_2; \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)\}$ and also $w(t_1, t_2) \geq 0$ is a measurable function satisfying

$$\int_0^1 \int_0^1 w(t_1, t_2) dt_1 dt_2 < \infty. \tag{21}$$

The above condition implies that the test statistic $T^{(.)}_{n,w}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$ is finite for the fixed sample size n. Similarly for the sub-model I & II with appropriate p.g.f. one can derive test statistic $T^{(SI)}_{P,n,w}$ and $T^{(SII)}_{P,n,w}$.

Due to the difficulty in obtaining an explicit expression for the critical region, it has been argued in Muñoz and Gamero [20] and in Muñoz [19], the rejection regions can be simulated using bootstrapping methods. The general procedure to identify an appropriate weight function is difficult to argue. One can consider the weight functions, which include a more prominent family of functions. A few weight functions are considered in Appendix C and also derived its test statistic. In Section 4, we analyzed the effect of weight functions and their feasible parameter values on the critical region.

3.2.2. Kocherlakota and Kocherlakota (K&K) method

Let $\mathbf{Z}_1, ..., \mathbf{Z}_n$ be a random sample from the bivariate distribution $F(\mathbf{z}; \underline{\theta})$, where $\underline{\theta} = (\theta_1, ..., \theta_d)^T$ is the d-dimensional parameter vector. Let $G(t_1, t_2; \underline{\theta})$ be the p.g.f. of $\mathbf{Z} = (X, Y)^T$, $t_1, t_2 \in \mathbb{R}^2$ and parameter vector $\underline{\theta}$ is estimated by the maximum likelihood estimation (m.l.e.) method, and the estimator we denote by $\widehat{\underline{\theta}}$. Let $G_n(t_1, t_2) = \frac{1}{n} \sum_{i=1}^n t_1^{X_i} t_2^{Y_i}$, $t \in \mathbb{R}$ be the empirical probability generating function (e.p.g.f.), then the test statistic is given by

$$T_N(t_1, t_2) = \frac{G_n(t_1, t_2) - G(t_1, t_2; \widehat{\underline{\theta}})}{\sigma}, \ |t_1| < 1; |t_2| < 1, \tag{22}$$

is asymptotically follows the standard normal distribution, where $\sigma^2 = \frac{1}{n}[G(t_1^2, t_2^2; \underline{\theta}) - G^2(t_1, t_2; \underline{\theta}))] - \sum_{i=1}^d \sum_{j=1}^d \sigma_{i,j} \frac{\partial G(t_1,t_2;\underline{\theta})}{\partial \theta_i} \frac{\partial G(t_1,t_2;\underline{\theta})}{\partial \theta_j}$, $((\sigma_{i,j}))$ is the inverse of the Fisher information matrix and σ can be estimated by plugging in the m.l.e. of $\underline{\theta}$. We refer to Kocherlakota and Kocherlakota (K&K) [12] for the asymptotic distribution of the test statistic. Note that the Fisher information matrix computation for the full model is theoretically cumbersome, yet one can use numerical methods to evaluate the matrix. However, we are considering the two sub-models of the bivariate pseudo-Poisson and deriving their test statistics.

Now, for the sub-model I, the Fisher information matrix is

$$I^{(SI)}(\lambda_1, \lambda_3) = n \begin{bmatrix} E\left(\frac{X}{\lambda_1^2}\right) & 0 \\ 0 & E\left(\frac{Y}{\lambda_3^2}\right) \end{bmatrix} = \begin{bmatrix} \frac{n}{\lambda_1} & 0 \\ 0 & \frac{n(1+\lambda_1)}{\lambda_3} \end{bmatrix}.$$

Similarly, for the sub-model II, the Fisher information matrix is

$$I^{(SII)}(\lambda_1, \lambda_3) = n \begin{bmatrix} E\left(\frac{X}{\lambda_1^2}\right) & 0 \\ 0 & E\left(\frac{Y}{\lambda_3^2}\right) \end{bmatrix} = \begin{bmatrix} \frac{n}{\lambda_1} & 0 \\ 0 & \frac{n\lambda_1}{\lambda_3} \end{bmatrix}.$$

The GoF test statistic is

$$T_{PN}^{(SI)}(t_1,t_2) = \frac{G_n(t_1,t_2) - G_I(t_1,t_2;\hat{\lambda_1},\hat{\lambda_3})}{\sigma^{(SI)}},\ |t_1|<1, |t_2|<1, \tag{23}$$

where $G_n(.)$ is empirical p.g.f. and $G_I(t_1,t_2;\hat{\lambda_1},\hat{\lambda_3})$ is estimated p.g.f. of the sub-model I, and

$$\begin{aligned}\sigma^{2(SI)} &= \frac{1}{n}[G_I(t_1^2,t_2^2;\lambda_1,\lambda_3) - G_I^2(t_1,t_2;\lambda_1,\lambda_3)] - \frac{\lambda_1}{n}\frac{\partial^2 G_I(t_1,t_2;\lambda_1,\lambda_3)}{\partial\lambda_1^2}\\ &\quad -\frac{\lambda_3}{n(\lambda_1+1)}\frac{\partial^2 G_I(t_1,t_2;\lambda_1,\lambda_3)}{\partial\lambda_3^2}.\end{aligned} \tag{24}$$

Similarly, for the sub-model II, the GoF test statistic will be

$$T_{PN}^{(SII)} = \frac{G_n(t_1,t_2) - G_{II}(t_1,t_2;\hat{\lambda_1},\hat{\lambda_3})}{\sigma^{(SII)}},\ |t_1|<1, |t_2|<1, \tag{25}$$

where $G_n(.)$ is empirical p.g.f. and $G_I(t_1,t_2;\hat{\lambda_1},\hat{\lambda_3})$ is estimated p.g.f. of the sub-model II, and

$$\begin{aligned}\sigma^{2(SII)}(t_1,t_2) &= \frac{1}{n}[G(t_1^2,t_2^2;\lambda_1,\lambda_3) - G_{(II)}^2(t_1,t_2;\lambda_1,\lambda_3)] - \frac{\lambda_1}{n}\frac{\partial^2 G_{(II)}(t_1,t_2;\lambda_1,\lambda_3)}{\partial\lambda_1^2}\\ &\quad -\frac{\lambda_3}{n\lambda_1}\frac{\partial^2 G_{II}(t_1,t_2;\lambda_1,\lambda_3)}{\partial\lambda_3^2}.\end{aligned} \tag{26}$$

The bootstrapped finite sample and asymptotic distributions of the GoF test statistic of $T_{PN}^{(.)}$ are studied in Section 4.

In the following, we propose a test procedure which will be Supremum on the absolute value of the K&K test statistic with (t_1,t_2) over $(-1,1)\times(-1,1)$. The reason behind proposing such a test is exemplified in Section 4. The mentioned GoF testing procedure for the K&K method is originally discussed in Feiyan Chen [6] for the univariate and bivariate geometric models. Besides, Feiyan Chen [6] also discusses the K&K method for the multiple t-values for the GoF test for geometric models, c.f. Page 12 of Chen[6] . However, in the present note, we are interested in proposing tests that are free from the choices of t-values; hence, the advantages or disadvantages of considering multiple t-values are not discussed or illustrated in this note.

The GoF test statistic is

$$T_{SPN}^{(.)} = \sup_{(t_1,t_2)\in\{(-1,1)\times(-1,1)\}}\left|\frac{G_n(t_1,t_2) - G_.(t_1,t_2;\hat{\lambda_1},\hat{\lambda_3})}{\sigma^{(.)}}\right|, \tag{27}$$

where $G_n(.)$, $G_.$ and $\sigma^{(.)}$ are defined in equation (22) to (25). Also note that deriving the asymptotic distribution of the statistic $T_{SPN}^{(.)}$ is theoretically ambiguous. Hence, in Section 4 the finite sample distribution of the test statistic $T^{(.)}_{SPN}$ is analyzed.

Remark 4. For the Muñoz and Gamero (M&G) and Kocherlakota and Kocherlakota(K&K) the estimated p.g.f.'s can be obtained by plugging in the m.l.e. estimates of λ_i, $i = 1, 2, 3$.

Remark 5. In Meintanis [15] Theorem 2.3 characterizes the general p.g.f. class of distributions called the CP-class, where Thomas distribution is a member of the CP-class. Which suggests a GoF test construction for the bivariate count models through the one-dimensional p.g.f. of particular form belongs to the CP-class, see Meintanis [15] Page 23 − 25. However, we have identified a missing link in the theorem, and it needs to be clearly justified that knowing the form of the one-dimensional p.g.f. assures in identifying the bivariate p.g.f.. If we assume the theorem, a family of tests can be generated for the bivariate Poisson-Poisson distribution by testing only the one-dimensional Thomas distribution. Since we do not completely agree with the theorem, we are not recommending any test based on the one-dimensional result to conclude on the higher dimensional tests.

3.3. GoF test free from alternative

In the class of distribution-free tests, the χ^2 test is commonly used even when there is no specific alternative hypothesis. However, this also raises difficulties in assessing the power of the test.

3.3.1. χ^2 GoF

In the following, we are using the classical χ^2 GoF test, and cell probabilities are computed up to k. The cell probability matrix is given by

$X — Y$	0	1	2	3	...	k+
0	p_{00}	p_{01}	p_{02}	p_{03}	...	$P(X=0)-\sum_{j=0}^{k-1} p_{0j}$
1	p_{10}	p_{11}	p_{12}	p_{13}	...	$P(X=1)-\sum_{j=0}^{k-1} p_{1j}$
2	p_{20}	p_{21}	p_{22}	p_{23}	...	$P(X=2)-\sum_{j=0}^{k-1} p_{2j}$
3	p_{30}	p_{31}	p_{32}	p_{33}	...	$P(X=3)-\sum_{j=0}^{k-1} p_{3j}$
...	...	...	...	...	...	...
k+	$P(Y=0)-\sum_{i=0}^{k-1} p_{i0}$	$P(Y=1)-\sum_{i=0}^{k-1} p_{i1}$	$P(Y=2)-\sum_{i=0}^{k-1} p_{i2}$	$P(Y=3)-\sum_{i=0}^{k-1} p_{i3}$	...	$1-\sum_{i=k}^{\infty}\sum_{j=k}^{\infty} p_{ij}$

where $p_{ij} = P(X = i, Y = j)$. The test statistic is

$$T_{\chi^2} = \sum_{i=0}^{k}\sum_{j=0}^{k} \frac{\Big(O_{i,j} - E_{i,j}\Big)^2}{E_{i,j}}, \tag{28}$$

where k is the truncation point, $O_{i,j}$ is frequency of (i, j) observation in the data of size n and $E_{i,j} = nP(X = i, Y = j)$. Hence, with Pearson theorem T_{χ^2} follows a χ^2 distribution with $[(k + 1) \times (k + 1) - 1 - 3]$ degrees of freedom.

Similarly, above two tests for the sub-models I & II can be derived with appropriate cell probabilities $p_{ij} = P(X = i, Y = j)$. In Section 4, we analyse finite sample and large sample behavior of the above two test statistics.

4. Examples

4.1. Simulation

In the following we give a general procedure to analyse the finite sample distribution of the GoF test statistics with bootstrapping technique.

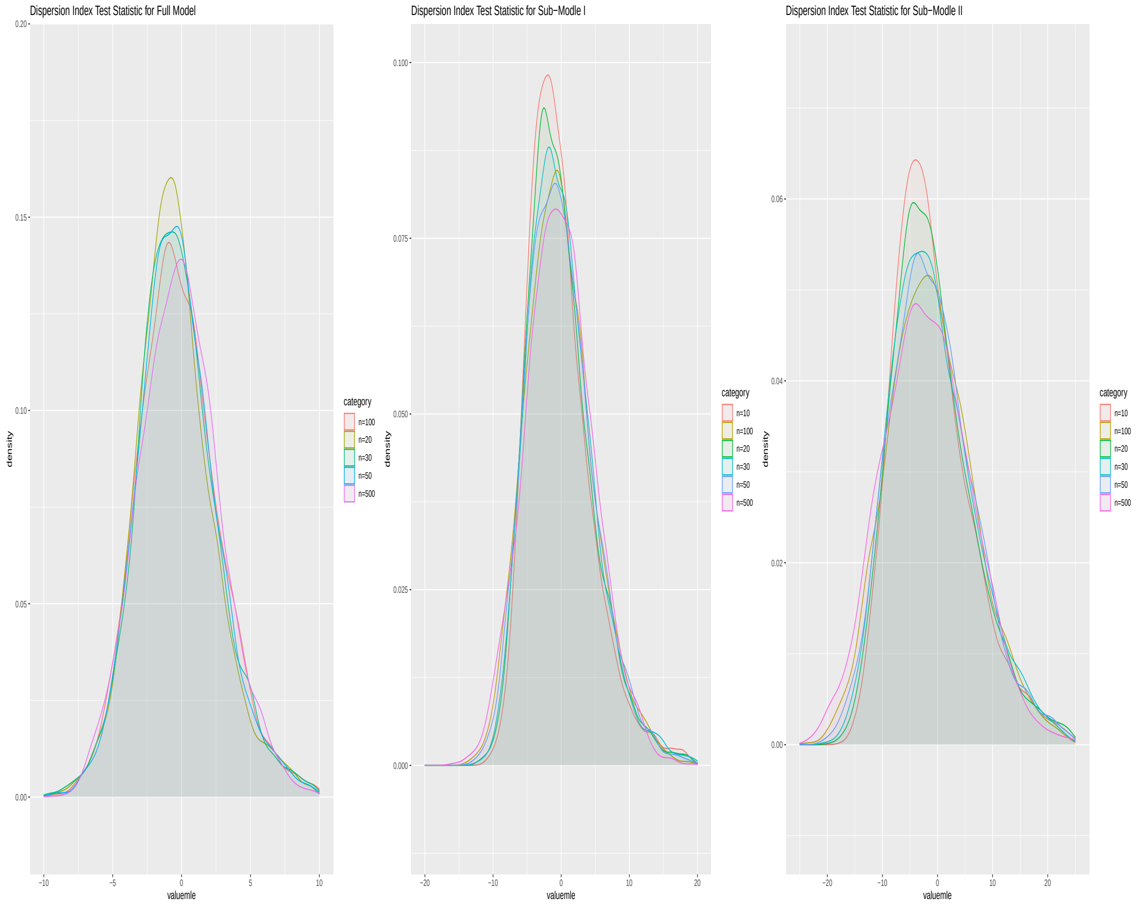
Step 1 Simulate n observations from the bivariate pseudo-Poisson with fixed parameter values. Otherwise, estimate parameters by moment or m.l.e. method, say $\hat{\lambda}_i$. Then compute GoF test statistics, say T_{obs}.

Step 2 Fix the number of bootstrapping samples, say B (ideal size is 5000,10000) and sample $m(< n)$ observation from the above sample, repeat Step 1 and compute $T^b_{m,obs}$ for $b \in \{1, 2, ..., B\}$.

Step 3 From the frequency distribution of $T^b_{m,obs}$ obtain the quantile values and the empirical p-value is $\frac{1}{B}\{\text{Total no. of } T^b_{m,obs} \text{ greater than } T_{obs}\}$.

4.1.1. Test based on moments

In the current section we will be analysing the new non-consistent test defined in Section 3.1. The finite sample distribution of the $FI_n^{(\cdot)}$, see Table 4 and Figure 1 for the distribution and its quantile values for the full and its sub-models. From the simulation study it clearly shown that the distribution of test statistic shown to be standard normal behaviour for increasing sample size. In addition, we make a note that for small and moderately large sample sizes the test shown to be stable and consistent.



(a) Full Model (b) Sub-Model I (c) Sub-Model II

Figure 1. Distribution of the $FI_n^{(.)}$

4.1.2. Muñoz and Gamero (M&G) method

Now, we consider GoF using p.g.f. (c.f. Muñoz and Gamero [20]) $T_{P,n,w}^{(.)}$ with depends on the underlying weight functions. We refer to Table 1, 2, 3 and Figure 2, 3, 4, 5, 6 for small and large sample distribution of the test statistic and its quantile values for the full and its sub-models.

To better understand the behaviour of the test statistic, we examined the impact of different weights at $a_1 = -0.9, -0.5, -0.01, 0.5, 3$ and $a_2 = -0.9, -0.5, -0.01, 0.5, 5$ on the test statistic. According to the simulation study we make a remark that irrespective of the weight chosen the test are consistent and stable for moderately large sample sizes. Also, note that for the increasing sample size the distribution of the test statistic are less variant and are shown to be consistent.

Table 1. Example 1

		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)			
		$n = 20$	$n = 30$	$n = 50$	$n = 100$
$T_{P,n,w}$	Full Model	(7.085, 8.664, 9.632; 30.382, 33.138, 38.773)	(5.117, 5.793, 6.190; 15.599, 16.820, 20.041)	(2.666, 3.002, 3.230; 6.908, 7.490, 8.600)	(0.885, 0.960, 1.005; 1.635, 1.720, 1.911)
	Sub Model I	(10.808, 12.772, 13.980; 40.614, 42.949, 49, 598)	(6.560, 7.770, 8.455; 21.579, 23.463, 28.388)	(3.535, 4.008, 4.009, 4.346; 8.914, 9.507, 10.722)	(0.958, 1.07, 1.126; 2.093, 2.217, 2.513)
	Sub Model II	(21.270, 29.847, 35.064; 147.186, 164.792, 197.586)	(16.4633, 21.968, 24.682; 78.691, 86.521, 103.263)	(12.906, 15.543, 16.812; 33.672, 36.303, 41.303)	(3.191, 3.582, 3.951; 7.981, 8.418, 9.304)

Table 2. Example 2

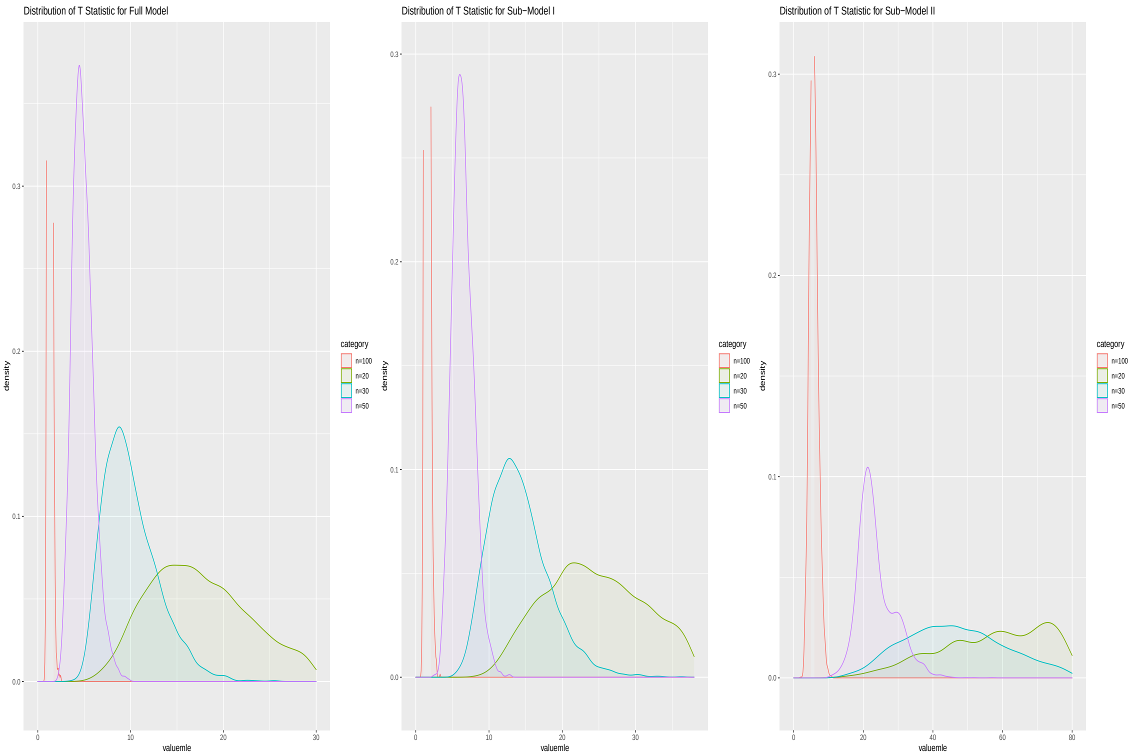
		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)			
		$n = 20$	$n = 30$	$n = 50$	$n = 100$
$T_{P,n,w}$	Full Model	(24.003, 29.695, 32.916; 109.461, 120.825, 143.666)	(18.688, 21.358, 23.045; 56.043, 60.840, 72.928)	(9.814, 11.380, 12.424; 18.230, 27.670, 33.967)	(3.340, 3.758, 3.964; 6.958, 7.362, 8.115)
	Sub Model I	(35.178, 41.764, 46.663; 153.635, 166.724, 205.881)	(21.55443, 26.233, 28.611, 81.378, 90.566, 111.364)	(12.925, 14.971, 16.330; 36.434, 39.121, 44.605)	(3.735, 4.290, 4.546; 9.0899.648, 11.126)
	Sub Model II	(83.950, 120.662, 150.705; 750.7120, 854.663, 1027.723)	(66.689, 94.637, 108.678; 398.689, 44.015, 540523)	(51.414, 64.302, 72.087; 169.210, 184.953, 213.675)	(13.119, 15.157, 16.856; 38.858, 41.334, 46.675)

Table 3. $T^{(.)}_{P,n,w}$

		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)			
		$n = 20$	$n = 30$	$n = 50$	$n = 100$
$T_{P,n,w}$	$(a_1 = -0.9, a_1 = -0.9)$	(30.918, 43.773, 54.367; 455.914, 527.123, 1023.20345)	(25.182, 33.193, 38.630; 204.290, 414.503)	(13.956, 17.766, 20.112; 85.184, 103.305, 140.462)	(4.543, 5.389, 5.885; 18.364, 22.210, 29.855)
	$(a_1 = -0.01, a_1 = -0.01)$	(12.663, 18.047, 19.808; 69.013, 77.235, 97.099)	(9.193, 11.206, 12.488; 36.494, 39.859, 47.084)	(4.981, 5.839, 6.373; 15.345, 16.561, 19.420)	(1.774, 1.951, 2.060; 3.755, 3.994, 4.461)
	$(a_1 = 0.01, a_1 = 0.01)$	(10.923, 13.726, 15.786; 54.873, 61.512, 75.468)	(7.864, 9.473, 10.465; 29.302, 31.877, 37.684)	(4.216, 4.899, 5.315, 12.289, 13.215, 15.395)	(1.506, 1.645, 1.734; 3.037, 3.213, 3.570)
	$(a_1 = 0.5, a_1 = 0.5)$	(8.400, 10.410, 11.925; 40.708, 45.527, 54.879)	(5.935, 7.091, 7.752; 20.261, 21.926, 25.859)	(3.122, 3.598, 3.884; 8.470, 9.118, 10.522)	(1.120, 1.213, 1.271; 2.100, 2.210, 2.445)
	$(a_1 = 3, a_1 = 5)$	(3.066, 3.557, 3.851; 10.352, 11.398, 13.679)	(3.032, 3.353, 3.566; 6.451, 6.745, 7.330)	(1.480, 1.618, 1.677; 2.695, 2.832, 3.089)	(0.235, 0.259, 0.365; 0.533, 0.551, 0.584)
	$(a_1 = -0.9, a_1 = 5)$	(16.295, 26.760, 36.107; 235.847, 262.617, 326.237)	(14.523, 22.047, 27.541, 124.208, 137.808, 165.083)	(0.213, 12.853, 15.190; 52.653, 57.075, 67.911)	(8.825, 0.945, 1.028; 8.050, 9.502, 12.366)

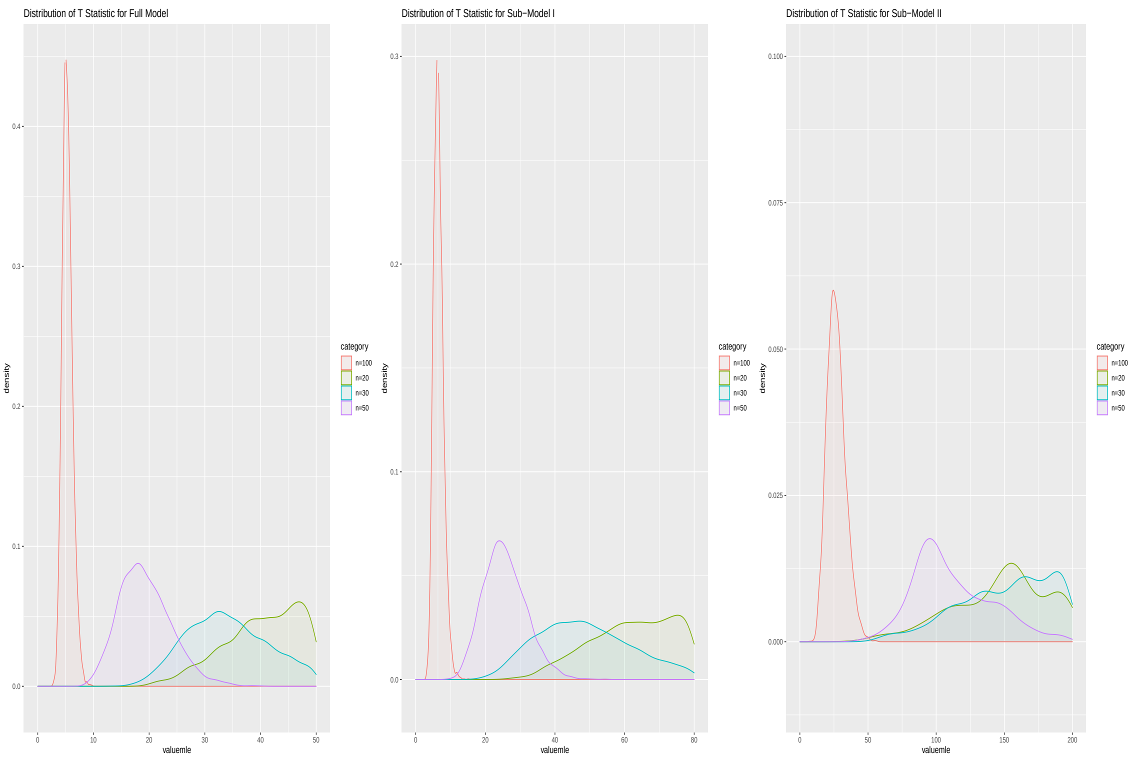
Table 4. Distribution of the $FI^{(.)}_n$

		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)				
		$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$FI^{(.)}_n$	Full Model	(−7.818, −5.701, −4.665; 5.033, 6.900, 11.231)	(−7.760, −5.762, −4.698; 5.084, 6.703, 10.598)	(−7.166, −5.530, −4.748; 5.034, 6.544, 9.666)	(−7.167, −5.530, −4.748; 5.034, 6.544, 9.666)	(−6.897, −5.676, −4.841; 5.047, 6.063, 8.20)
	Sub Model I	(−9.537, −7.749, −6.766; 9.093, 11.852, 17.648)	(−9.947, −8.140, −7.131; 8.825, 11.500, 15.842)	(−10.477, −8.560, −7.505; 8.725, 10.887, 15.936)	(−11.018, −8.610, −7.507; 9.203, 11.223, 15.689)	(−11.759, −9.034, −7.802; 8.365, 10.293, 13.741)
	Sub Model II	(−15.205, −12.628, −11.181; 13.710, 18.740, 29.800)	(−15.917, −13.277, −11.695; 13.924, 17.255, 28.748)	(−17.191, −14.088, −12.162; 13.748, 17.947, 27.665)	(−18.434, −15.108, −13.076; 12.997, 15.976, 21.976)	(−20.647, −16.782, −14.478, 11.629, 14.497, 21.093)



(a) (b) (c)

Figure 2. Example 1



(a) (b) (c)

Figure 3. Example 2

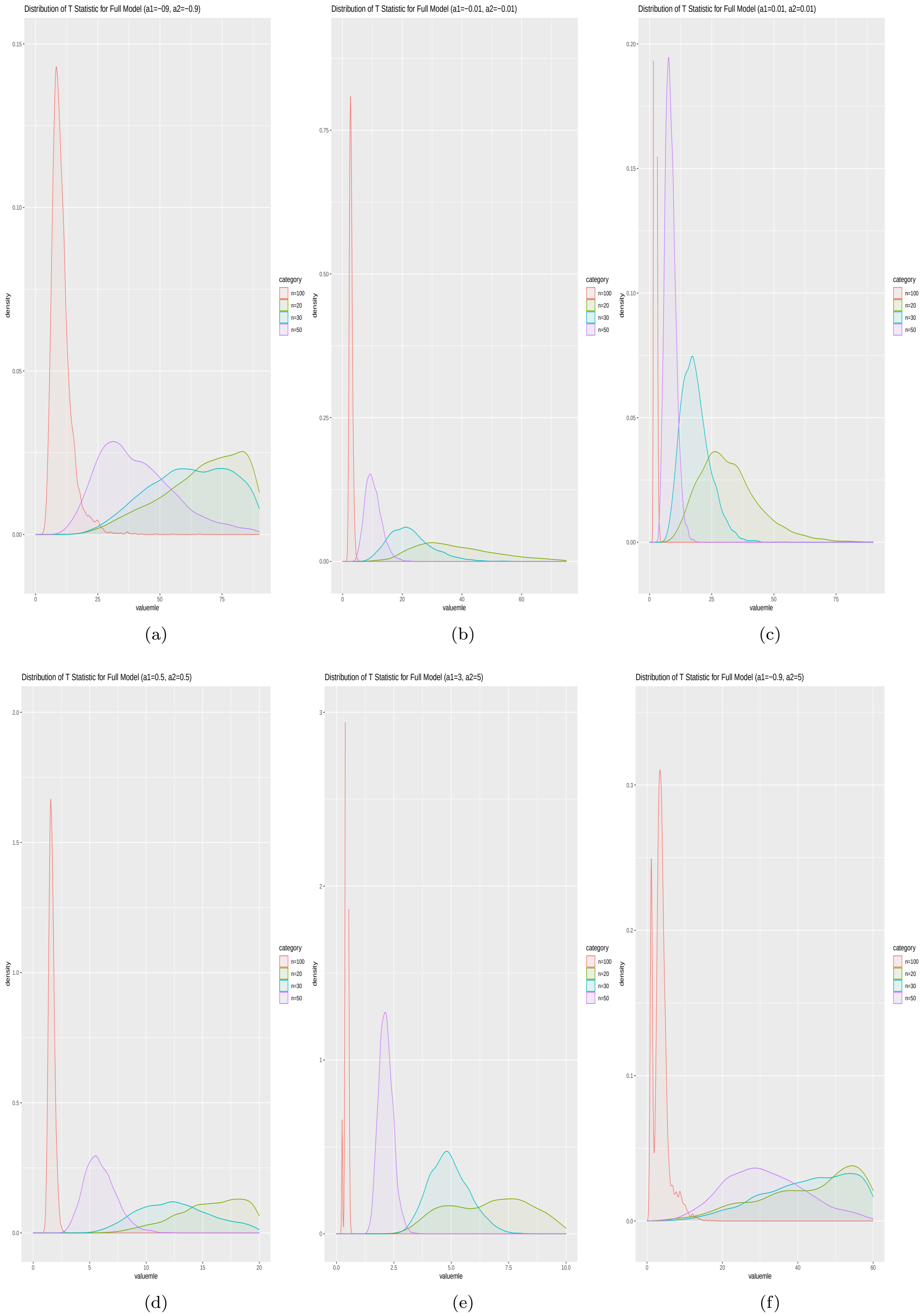


Figure 4. $T^{(.)}_{P,n,w}$

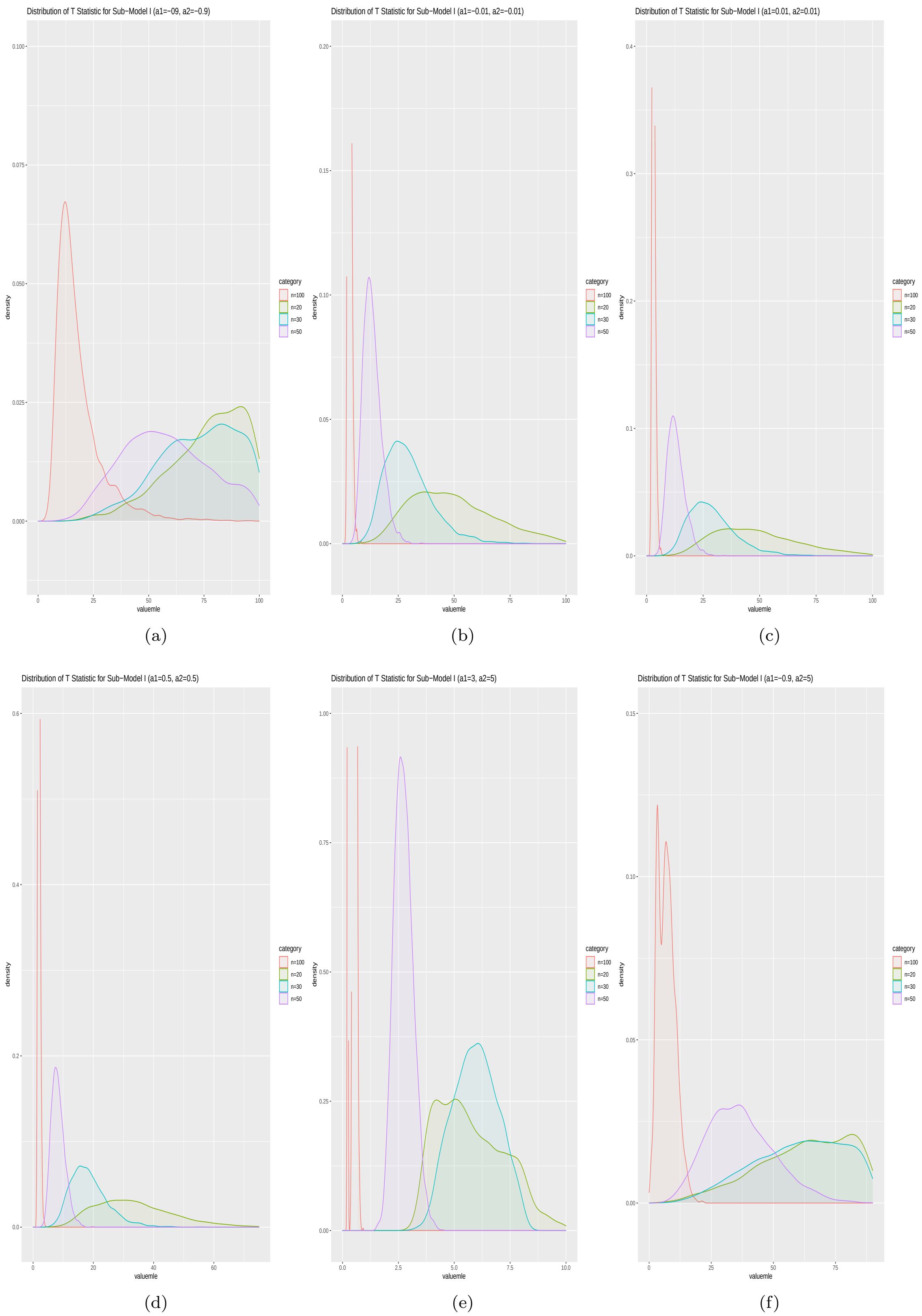
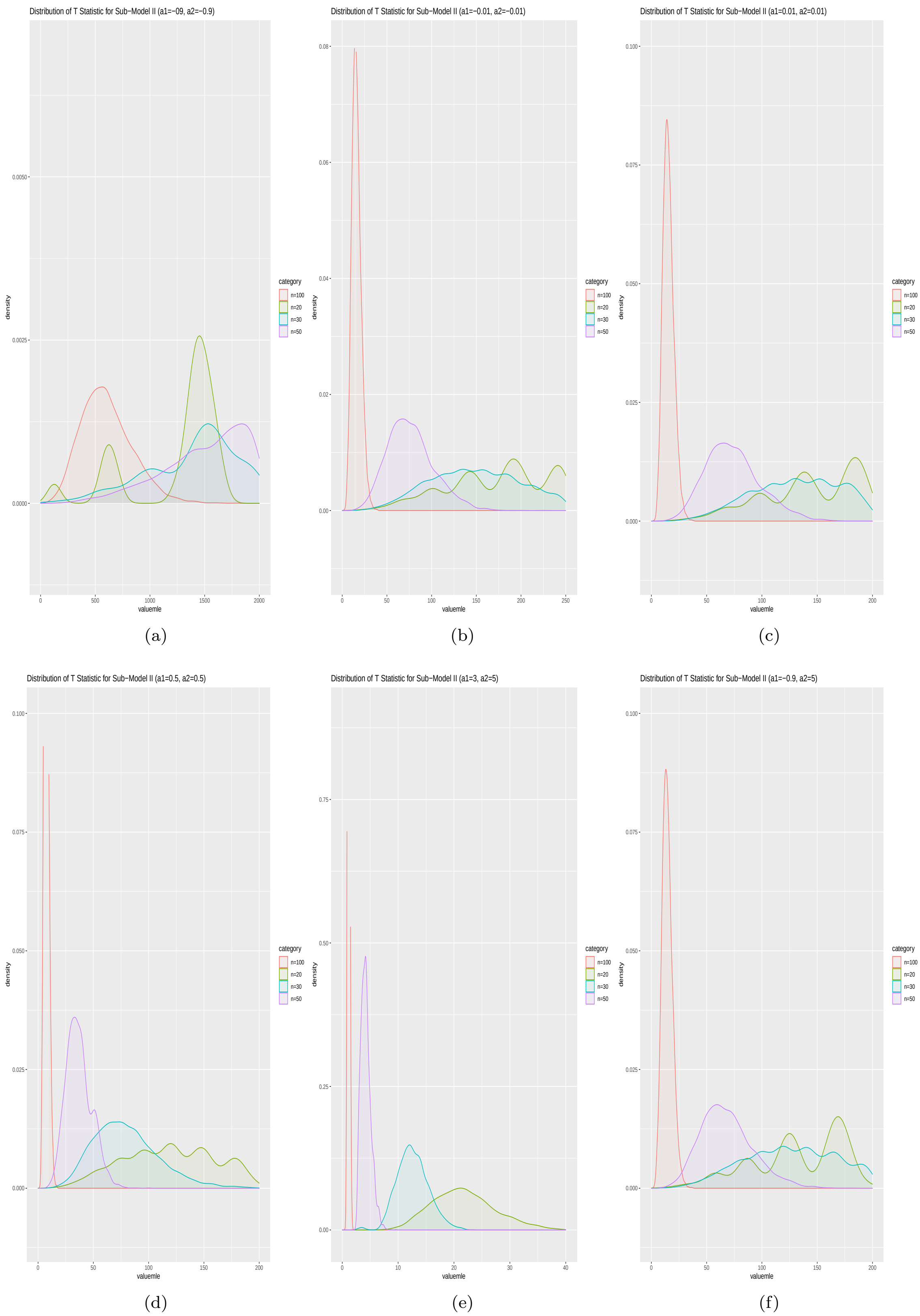


Figure 5. $T^{(SI)}_{P,n,w}$



(a) (b) (c)

(d) (e) (f)

Figure 6. $T^{(SII)}_{P,n,w}$

4.1.3. K&K Method

In the following, we discuss finite, large and asymptotic distribution of the test statistics $T^{(.)}_{PN}(t_1, t_2)$ and $T^{(.)}_{SPN}$ (c.f. Section 3.2.2). Here, we limit our analysis to sub-models of the bivariate pseudo-Poisson model, and statistical inference or parameter estimation are well defined (see Section 7 in Arnold and Manjunath [2]). However, in sub-models I and II, both the method of moments and the maximum likelihood estimators coincides. Hence, due to invariance property of the maximum likelihood estimator the defined test statistic asymptotically follows standard normal with variance is will be inverse of Fisher information matrix.

Now, we consider bootstrapping size of $B = 5000$ with varying sample size of $n = 20, 30, 50, 100, 500$ at different $t_i = \pm 0.01, \pm 0.5, \pm 0.9$, $i = 1, 2$.

Sub-Model I (i.e. $\lambda_2 = \lambda_3$) The corresponding quantile values and density plots refer to Table 5 and Figure 7, respectively.

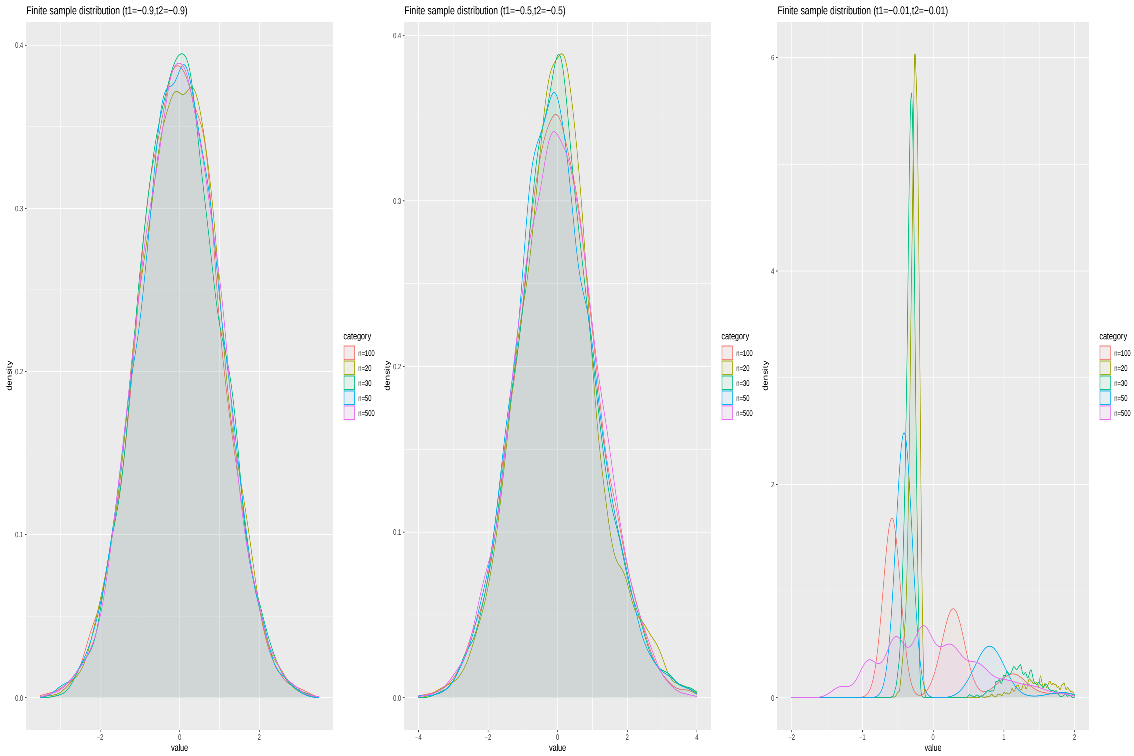
Sub-Model II (i.e. $\lambda_2 = 0$) The corresponding quantile values and density plots refer to Table 6 and Figure 8, respectively.

According to the simulation study it has been observed that whenever t_i is closer to zero the empirical critical points are closer to the standard normal quantile values. It has been recommended that the t values are to be chosen either in the neighbourhood of zero or well spanned in the interval $(-1, 1)$ to have consistency in the tests.

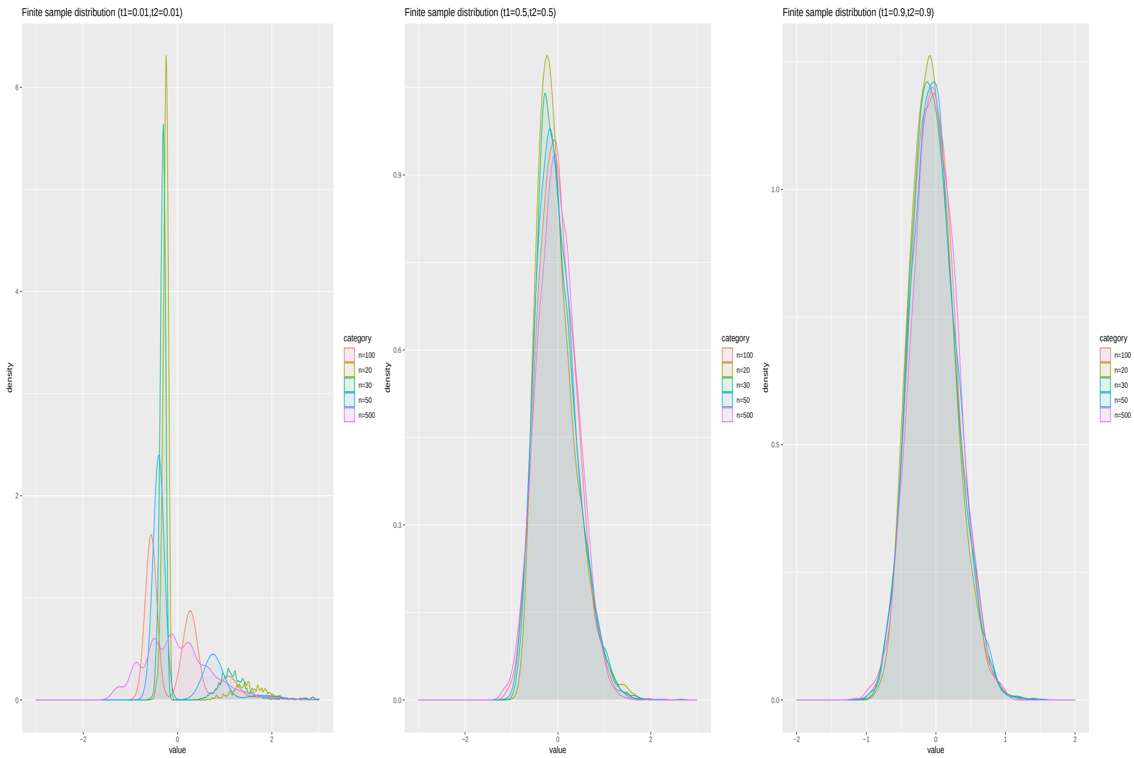
Note that from the Table 5 & 6 and also from Figure 7 & 8 K&K method finite sample distribution depends on the selected values for (t_1, t_2). In particular, at $t_1 = -0.5(0.5)$ and $t_2 = -0.5(0.5)$ K&K statistic distributions are inconsistent. Hence, we consider the test statistic $T^{(.)}_{SPN}$(defined in (27)) such that the test support completely depends on complete span of t_i-values, $i = 1, 2,$. For an illustration of the proposed test we are analysing the finite sample distribution of the test statistic which is computed with varying t_1 and t_2 from -0.99 to 0.99 at an increment of 0.01.

Finally, it has been argued in Feiyan Chen [6] that such tests are robust to the choice of alternatives and that the performance of the test is better than the K&K test because it also spanned the entire interval of $(-1, 1) \times (-1, 1)$.

We refer to Table 7 and Figure 9 for the quantile values and frequency distribution of the test statistic, respectively. The test statistic's behaviour is more stable and consistent for small and moderately large samples.

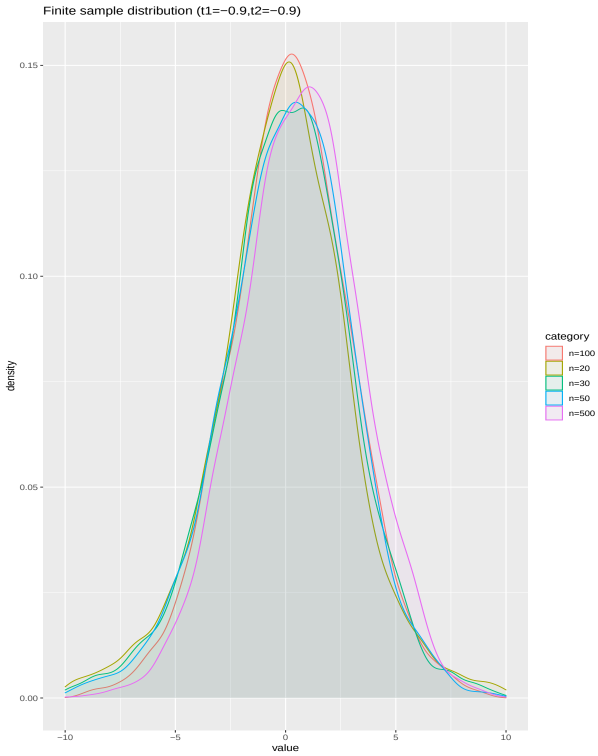


(a) $t_1 = -0.9, t_2 = -0.9$ (b) $t_1 = -0.5, t_2 = -0.5$ (c) $t_1 = -0.01, t_2 = -0.01$

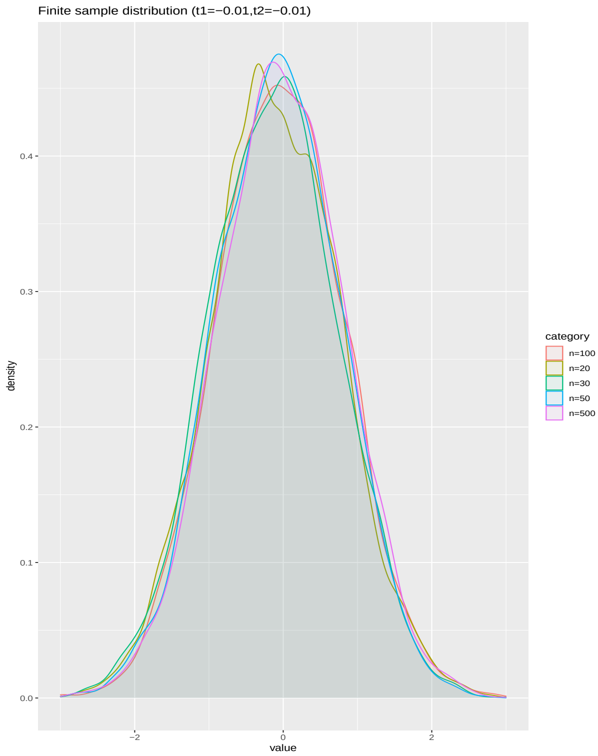


(d) $t_1 = 0.01, t_2 = 0.01$ (e) $t_1 = 0.5, t_2 = 0.5$ (f) $t_1 = 0.9, t_2 = 0.9$

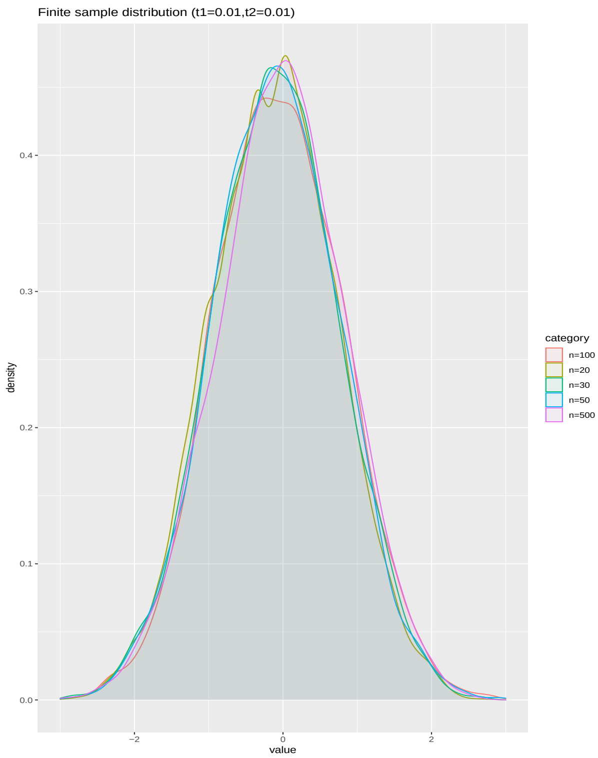
Figure 7. Finite sample distribution of $T_{PN}^{(SI)}$ for the Sub-Model I



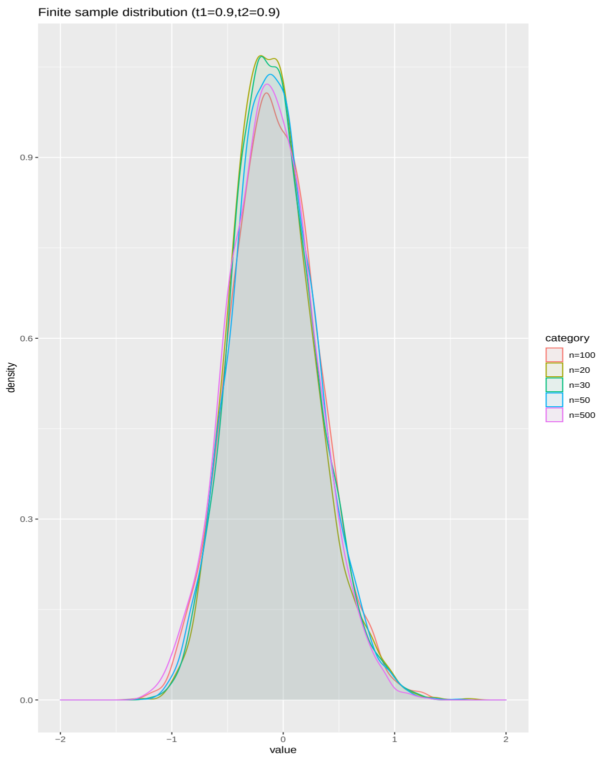
(a) $t_1 = -0.9, t_2 = -0.9$



(b) $t_1 = -0.01, t_2 = -0.01$

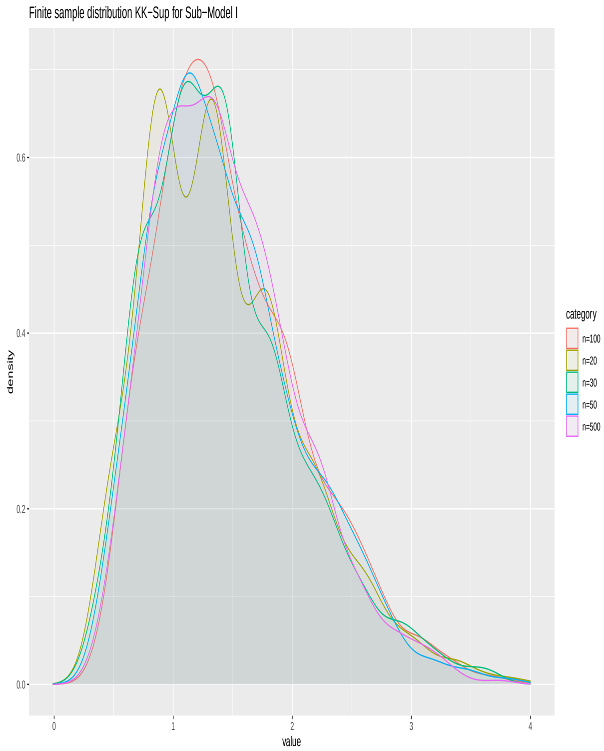


(c) $t_1 = 0.01, t_2 = 0.01$

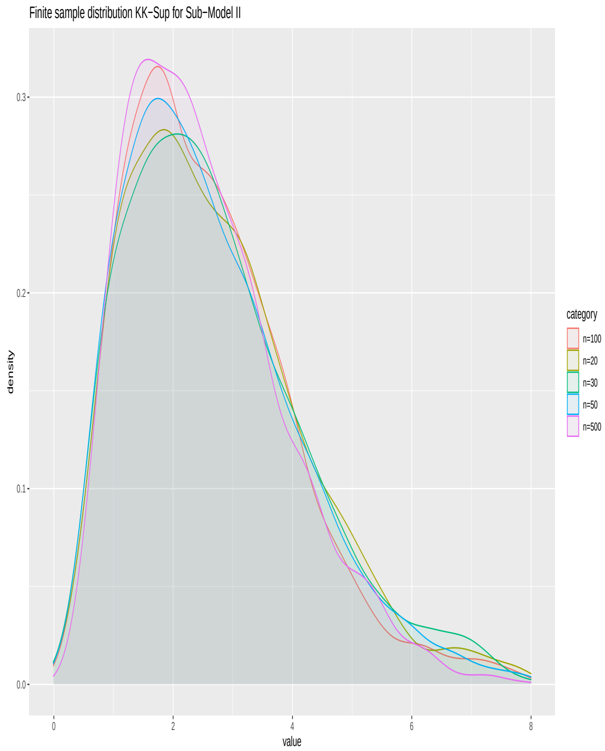


(d) $t_1 = 0.9, t_2 = 0.9$

Figure 8. Finite sample distribution of $T_{PN}^{(SII)}$ for the Sub-Model II



(a) $T_{SPN}^{(SI)}$



(b) $T_{SPN}^{(SII)}$

Figure 9. Finite sample distribution of the supremum of absolute deviation for the K&K-test statistic.

4.1.4. GoF test free from alternative

The distribution of Chi-square GoF test statistic sample distribution for the full and its sub-models, see Figure 10. However, in the case of missing alternative distribution information Chi-square GoF test recommended otherwise other tests which are mentioned perform better than the Chi-square. Also, Chi-square test dependences on the value of k chosen, for an illustration we have consider $K = 4$ and analysed its finite and large sample distributions.

Table 5. $T_{PN}^{(SI)}$ distribution for the Sub-Model I

		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)				
		$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$T_{PN}^{(SI)}$	$t_1 = -0.9$, $t_2 = -0.9$	(−2.503, −1.953, −1.652; 1.660, 1.912, 2.522)	(−2.447, −1.886, −1.626; 1.637, 1.981, 2.528)	(−2.582, −1.975, −1.669; 1.619, 1.928, 2.548)	(−2, 622, −2.011, −1.674; 1.620, 1.912, 2.543)	(−2.582, −1.910, −1.612; 1.634, 1.948, 2.560)
	$t_1 = -0.5$, $t_2 = -0.5$	(−2.839, −2.187, −1.811, 1.953, 2.458, 3.196)	(−2.986, −2.168, −1.825, 2.031, 2.523, 3.390)	(−2.917, −2.189, −1.825; 1.927, 2.412, 3.342)	(−2.822, −2.198, −1.872; 1.988, 2.364, 3.106)	(−3.021, −2.156, −1.845; 1.947, 2.270, 3.078)
	$t_1 = -0.01$, $t_2 = -0.01$	(−0.464, −0.402, −0.376; 1.701, 1.976, 2.981)	(−0.519, −0.456, −0.428; 1.454, 1.688, 2.912)	(−0.593, −0.535, −0.510; 1.407, 2.036, 2.936)	(−0.747, −0.695, −0.668; 1.332, 1.734, 2.453)	(−1.388, −1.263, −0.997; 1.228, 1.553, 2.220)
	$t_1 = 0.01$, $t_2 = 0.01$	(−0.452, −0.392, −0.362; 1.620, 1.870, 2.456)	(−0.492, −0.440, −0.415; 1.352, 1.612, 2.853)	(−0.574, −0.520, 0.492; 1.231, 1.867, 2.456)	(−0.713, −0.665, −0.641, 1.240, 1.645, 2.347)	(−1.310, −1.216, −0.958; 1.223, 1.529, 2.049)
	$t_1 = 0.5$, $t_2 = 0.5$	(−0.751, −0.640, −0.581; 0.771, 0.977, 1.423)	(−0.821, −0.692, −0.619; 0.760, 0.966, 1.281)	(−0.863, −0.725, −0.634; 0.753, 0.9358, 1.304)	(−0.944, −0.753, −0.654; 0.710, 0.892, 1.238)	(−1.008, −0.792, −0.683; 0.707, 0.853, 1.111)
	$t_1 = 0.9$, $t_2 = 0.9$	(−0.750, −0.624, −0.541; 0.498, 0.625, 0.880)	(−0.733, −0.622, −0.541; 0.516, 0.630, 0.854)	(−0.776, −0.625, −0.543; 0.545, 0.643, 0.884)	(−0.833, −0.657, −0.550; 0.539, 0.654, 0.867)	(−0.829, −0.648, −0.552; 0.554, 0.669, 0.842)

Table 6. $T_{PN}^{(SII)}$ distribution for the Sub-Model II

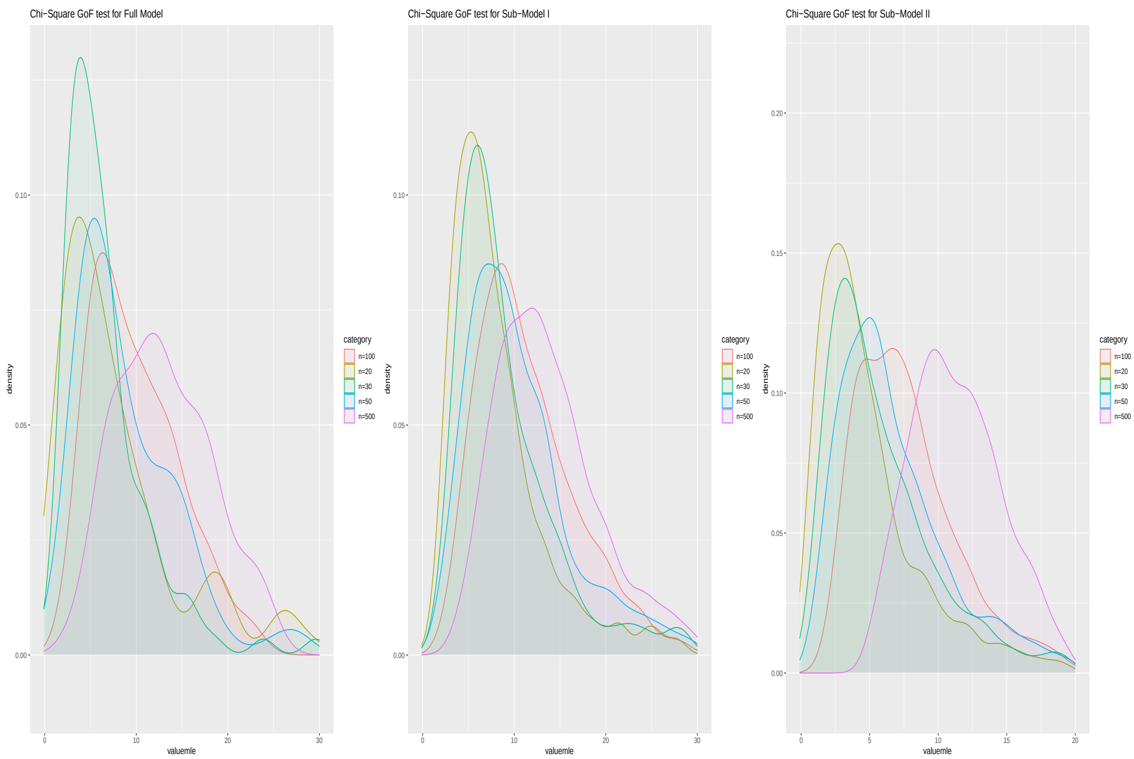
		Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)				
		$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$T_{PN}^{(SII)}$	$t_1 = -0.9$, $t_2 = -0.9$	(−17.388, −8.192, −5.956; 5.168, 7.046, 13.129)	(−13.272, −6.980, −5.286; 4.862, 5.972, 10.505)	(−9.952, −6.382, −5.020; 4.653, 5.737, 7.871)	(−7.590, −5.342, −4.345; 4.612, 5.568, 7.522)	(−6.667, −4.788, −3.7800; 5.200, 5.930, 7.723)
	$t_1 = -0.01$, $t_2 = -0.01$	(−2.306, −1.700, −1.557; 1.347, 1.608, 2.216)	(−2.222, −1.803, −1.507; 1.303, 1.587, 2.116)	(−2.241, −1.726, −1.393; 1.264, 1.517, 2.023)	(−2.174, −1.679, −1.441; 1.341, 1.624, 2.230)	(−2.203, −1.690, −1.390; 1.362, 1.588, 2.173)
	$t_1 = 0.01$, $t_2 = 0.01$	(−2.301, −1.740, −1.432; 1.203, 1.549, 2.007)	(−2.215, −1.802, −1.440; 1.297, 1.558, 1.965)	(−2.176, −1.733, −1.480; 1.259, 1.568, 2.123)	(−2.261, −1.680, −1.434; 1.383, 1.618, 2.136)	(−2.219, −1.727, −1.436; 1.363, 1.624, 2.065)
	$t_1 = 0.9$, $t_2 = 0.9$	(−0.927, −0.737, −0.647; 0.580, 0.750, 1.014)	(−0.913, −0.755, −0.657; 0.572, 0.714, 0.986)	(−0.950, −0.784, −0.669; 0.590, 0.723, 0.981)	(−0.980, −0.825, −0.701, 0.594, 0.748, 1.012)	(−1.033, −0.853, −0.730; 0.549, 0.672, 0.906)

Table 7. $T_{SPN}^{(SI)}$ distribution for the Sub-Model I

	Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)				
	$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$T_{SPN}^{(SI)}$	(0.380, 0.450, 0.528; 2.647, 2.987, 3.005)	(0.337, 0.489, 0.586; 2.656, 3.012, 3.622)	(0.401, 0.511, 0.593; 2.623, 2.863, 3.567)	(0.423, 0.574, 0.643; 2.679, 3.013, 3.591)	(0.410, 0.550, 0.634; 2.558, 2.860, 3.279)

Table 8. $T_{SPN}^{(SII)}$ distribution for the Sub-Model II

	Sample size (0.5%, 2.5%, 5%; 95%, 97.5%, 99.5%)				
	$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$T_{SPN}^{(SII)}$	(0.367, 0.676, 0.830; 6.868, 9.190, 24.875)	(0.422, 0.651, 0.771; 6.121, 7.016, 11.323)	(0.394, 0.6200.783; 5.559, 6.506, 9.039)	(0.423, 0.638, 0.793; 5.208, 6.221, 7.612)	(0.487, 0.701, 0.850; 5.113, 5.588, 6.720)



(a) Full Model (b) Sub-Model I (c) Sub-Model II

Figure 10. Chi-square GoF test for $k = 4$.

4.1.5. Power analysis

In the present section, we will be considering classical bivariate Poisson and bivariate Conway-Maxwell Poisson distributions as alternatives to analyse the power of each of the tests discussed above.

Hence, we simulate $n = 20, 30, 50, 100, 500$ samples from $Z_i \sim Poisson(\theta_i)$, $i = 1, 2, 3$ and taking $U = Z_1 + Z_3$ and $V = Z_2 + Z_3$ the resultant joint random variable (U, V) will be n observations from the classical bivariate Poisson distribution. Nevertheless, to simulate $n = 20, 30, 50, 100, 500$ samples from the bivariate Conway-Maxwell Poisson, we begin with simulating an observation from the univariate Conway-Maxwell Poisson with parameter θ and ν, say N. Further, simulate N observations from the bivariate binomial distribution with specified cell probabilities, say (W_{1i}, W_{2i}), $i = 1, 2, ..., N$. Then, the random vector $(\sum_{i=1}^{N} W_{1i}, \sum_{i=1}^{N} W_{2i})$ will be an observation from the bivariate Conway-Maxwell Poisson distribution. For the desired sample size, repeat the above procedure for n times to have specified sample size from the bivariate Conway-Maxwell Poisson distribution. We refer to Sellers et al. [23] for further discussion and an algorithm to simulate from the bivariate Conway-Maxwell Poisson using R software.

The empirical power computation is as follows

Step 1 Compute GoF test statistic value for the samples from alternative distribution, say T_{obs}.

Step 2 For the given bootstrapping size (say $B = 5000$), compute T_A^b for $b \in \{1, 2, ..., B\}$.

Step 3 Hence, $\frac{1}{B}\{$Total no. of T_A^b greater than $T_{obs}\}$ is an empirical power of the test.

We refer to Table 9 for the each of the tests empirical powers.

Table 9. Power (% of observations) under classical bivariate Poisson (BCBP$((\theta_1 = 1, \theta_2 = 3, \theta_3 = 4))$) and bivariate Conway-Maxwell Poisson (BCMP$(\theta = 1, \nu = 5,\ \mu_1 = 0.1, ratio = \exp(1.5))$) alternatives

		Sample size				
		$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$T_{PN}^{(SII)}$	$t_1 = -0.9,\ t_2 = -0.9$	$(17.2, 8.8)$	$(21.6, 4.2)$	$(22.6, 2.6)$	$(59.2, 0.03)$	$(90.4, 0.01)$
	$t_1 = -0.5,\ t_2 = -0.5$	$(0.92, 82.4)$	$(0.89, 84.0)$	$(0.91, 92.4)$	$(0.93, 0.95.8)$	$(0.99, 0.97)$
	$t_1 = -0.01,\ t_2 = -0.01$	$(0.99, 0.91)$	$(0.97, 0.93)$	$(0.99, 0.99)$	$(0.99, 0.97)$	$(0.99, 0.99)$
	$t_1 = 0.01,\ t_2 = 0.01$	$(0.99, 0.99)$	$(0.99, 0.98)$	$(0.98, 0.97)$	$(0.99, 0.99)$	$(0.99, 0.98)$
	$t_1 = 0.5,\ t_2 = 0.5$	$(0.99, 0.99)$	$(0.91, 0.99)$	$(0.92, 0.99)$	$(0.89, 0.99)$	$(0.81, 0.99)$
	$t_1 = 0.9,\ t_2 = 0.9$	$(0.98, 0.99)$	$(0.92, 0.93)$	$(0.94, 0.96)$	$(0.99, 0.99)$	$(0.99, 0.99)$
$T_{SPN}^{(SII)}$	—	$(0.95, 0.92)$	$(0.99, 0.91)$	$(0.95, 0.98)$	$(0.99, 0.99)$	$(0.99, 0.97)$
$T_{P,n,w}^{(\cdot)}$	$a_1 = -0.9, a_2 = -0.9$	$(35.2, 67.8)$	$(56.9, 82.7)$	$(59.9, 12.6)$	$(35.0, 9.2)$	−−
	$a_1 = -0.01,\ a_2 = -0.01$	$(6.0, 85.4)$	$(72.5, 27.4)$	$(24.5, 15.2)$	$(10.1, 50.3)$	−−
	$a_1 = 0.01,\ a_2 = 0.01$	$(79.5, 2.6)$	$(20.3, 16.0)$	$(36.3, 34.2)$	$(6.1, 59.8)$	−−
	$a_1 = 0.5,\ a_2 = 0.5$	$(19.2, 37.3)$	$(4.5, 27.0)$	$(26.1, 85.4)$	$(2.1, 55.5)$	−−
	$a_1 = 3,\ a_2 = 5$	$(35.0, 74.0)$	$(72.7, 19.0)$	$(5.5, 25.5)$	$(1.2, 10.0)$	−−
	$a_1 = -0.9,\ a_2 = 5$	$(12.0, 91.0)$	$(10.8, 72.0)$	$(4.7, 92.8)$	$(0.1, 43.6)$	−−
$T_{P,n,w}^{(SI)}$	$a_1 = -0.9, a_2 = -0.9$	$(79.3, 96.5)$	$(12.3, 95.0)$	$(11.2, 21.8)$	$(13.4, 34.6)$	−−
	$a_1 = -0.01,\ a_2 = -0.01$	$(73.5, 20.4)$	$(16.9, 9.5)$	$(0.9, 87.0)$	$(0.1, 13.2)$	−−
	$a_1 = 0.01,\ a_2 = 0.01$	$(22.4, 3.8)$	$(15.9, 96.4)$	$(2.2, 93.8)$	$(13.8, 17.4)$	−−
	$a_1 = 0.5,\ a_2 = 0.5$	$(43.2, 87.1)$	$(27.5, 4.8)$	$(21.3, 80.2)$	$(1.1, 69.4)$	−−
	$a_1 = 3,\ a_2 = 5$	$(40.8, 43.2)$	$(36.0, 94.7)$	$(2.1, 75.8)$	$(1.3, 10.2)$	−−
	$a_1 = -0.9,\ a_2 = 5$	$(54.4, 87.2)$	$(62.7, 72.0)$	$(4.7, 36.8)$	$(0.1, 24.4)$	−−
$T_{P,n,w}^{(SII)}$	$a_1 = -0.9, a_2 = -0.9$	$(7.1, 12.9)$	$(41.9, 99.9)$	$(7.7, 26.2)$	$(60.71, 64.2)$	−−
	$a_1 = -0.01,\ a_2 = -0.01$	$(71.4, 74.7)$	$(9.6, 56.6)$	$(2.9, 31.6)$	$(0.2, 36.5)$	−−
	$a_1 = 0.01,\ a_2 = 0.01$	$(8.1, 36.1)$	$(2.5, 54.4)$	$(7.5, 67.1)$	$(0.1, 37.3)$	−−
	$a_1 = 0.5,\ a_2 = 0.5$	$(38.9, 18.9)$	$(18.2, 99.6)$	$(5.4, 78.7)$	$(0.4, 96.2)$	−−
	$a_1 = 3,\ a_2 = 5$	$(2.6, 70.0)$	$(2.5, 1.8)$	$(0.1, 82.3)$	$(0.0, 6.0)$	−−
	$a_1 = -0.9,\ a_2 = 5$	$(50.6, 70.6)$	$(55.0, 36.4)$	$(4.4, 84.4)$	$(0.0, 13.2)$	−−
$FI_n^{(\cdot)}$	—	$(58.4, 34.6)$	$(46.1, 86.7)$	$(56.0, 22.9)$	$(11.8, 42.4)$	$(63.5, 20.8)$
$FI_n^{(SI)}$	—	$(76.9, 13.5)$	$(6.3, 90.6)$	$(83.0, 40.6)$	$(41.7, 30.9)$	$(88.1, 73.8)$
$FI_n^{(SII)}$	—	$(69.9, 94.4)$	$(86.4, 70.2)$	$(95.0, 26.6)$	$(91.0, 46.7)$	$(100, 33.8)$

All tests are effective or significant in identifying from the pseudo-Poisson and Conway-Maxwell Poisson distributions, according to the power analysis. When compared to the classical bivariate Poisson, tests are moderately consistent in detecting the true population. We draw the conclusion that one needs to think about altering the parameter values and conducting additional research on the same in order to better grasp the power for the classical bivariate Poisson alternative.

4.2. Real-life data

In the following section we consider two data sets which are mentioned in Karlis and Tsiamyrtzis [11], Islam and Chowdhury [9], Leiter and Hamdani [14] and also in Arnold and Manjunath [2]. For empirical p-value computation we have simulated 5000 observations from the pseudo-Poisson models with respective maximum likelihood values and compare it with the critical value of each of the tests.

4.3. A particular data set I

We consider a data sets which is mentioned in Islam and Chowdhury [9] and also in Arnold and Manjunath [2] , the source of the data is from the tenth wave of the Health and Retirement Study (HRS). The data represents the number of conditions ever had (X) as mentioned by the doctors and utilization of healthcare services (say, hospital, nursing home, doctor and home care) (Y). The Pearson correlation coefficient between X and Y is 0.063. The test for independence, classical inference (m.l.e and moment estimates) and AIC values for full and its sub-models (c.f. Arnold and Manjunath [2] page 16 and 18) in Table 10.

In the following we will consider the full and its sub-model II. The criteria of selecting below two models are discussed in Arnold and Manjunath [2] on page 18 and Table 10. We refer to Table 10 for the critical values and its empirical p-values for the Full and sub-model II.

Table 10. Health and retirement study data (Full Model) and m.l.e. estimates Full model ($\hat{\lambda_1} = 2.643, \hat{\lambda_2} = 0.688, \hat{\lambda}_3 = 0.031$) for Sub-Model II ($\hat{\lambda}_3 = 0.031$)

		$n = 5567$	
		Test statistic value	p-value
$T_{PN}^{(SII)}$	$t_1 = -0.9,\ t_2 = -0.9$	151.734	0.025
	$t_1 = -0.5,\ t_2 = -0.5$	-2870.383	0.891
	$t_1 = -0.01,\ t_2 = -0.01$	755.821	0.901
	$t_1 = 0.01,\ t_2 = 0.01$	803.119	0.921
	$t_1 = 0.5,\ t_2 = 0.5$	1713.7	0.141
	$t_1 = 0.9,\ t_2 = 0.9$	3710.615	0.164
$T_{SPN}^{(SII)}$	—	12.740	0.097
$T_{PN}^{(.)}$	$a_1 = -0.9,\ a_2 = -0.9$	578.674	0.01
	$a_1 = -0.5,\ a_2 = -0.55$	117.940	0.99
	$a_1 = -0.01,\ a_2 = -0.01$	64.179	0.8
	$a_1 = 1,\ a_2 = 1$	67.564	0.09
	$a_1 = 3,\ a_2 = 5$	21.739	0.12
	$a_1 = -0.9,\ a_2 = 5$	23.830	0.02
$T_{PN}^{(SII)}$	$a_1 = -0.9,\ a_2 = -0.9$	659.816	0.99
	$a_1 = 1,\ a_2 = 1$	71.881	0.07
	$a_1 = -0.9,\ a_2 = 5$	24.465	0.02
$FI_n^{(.)}$	—	-13.532	0.987
$FI_n^{(SII)}$	—	-25.729	0.991
Chi-square $^{(.)}$	—	417.653	$--$

The tests $T_{PN}^{(SII)}$ on neighbourhood of 0, $T_{SPN}^{(SII)}$, $T_{PN}^{(.)}$ (large than -0.9), $FI_n^{(.)}$ and $FI_n^{(SII)}$ are suggests that the Health and Retirement data fits bivariate pseudo-Poisson Full and its sub-model II, which agree with the AIC values listed on pages 16 & 18 of

Arnold and Manjunath's [2].

4.4. A particular data set II

Now, we consider a data set which is in Leiter and Hamdani [14], the source of the data is a 50-mile stretch of Interstate 95 in Prince William, Stafford and Spotsylvania counties in Eastern Virginia. The data represents the number of accidents categorized as fatal accidents, injury accidents or property damage accidents, along with the corresponding number of fatalities and injuries for the period 1 January 1969 to 31 October 1970. For classical inference (m.l.e. and moment estimates) and AIC values for full and its sub-models c.f. Arnold and Manjunath [2] page 17 and 19 (Table 11). The criteria of selecting below two models are discussed in Arnold and Manjunath [2] on page 19 and Table 11. It has been emphasized in Leiter and Hamdani [14] and Arnold and Manjunath [2] that mirrored sub-model II fit the data better than any other sub-models.

In the following we will consider the two models. We refer to Table 11 for the critical values and its empirical p-values for the Full and Mirrored sub-model II.

Table 11. Accidents and fatalities (Full Model) and m.l.e. estimates Full model ($\hat{\lambda_1} = 0.058$,$\hat{\lambda_2} = 0.812$,$\hat{\lambda}_3 = 0.867$) and for mirrored Sub-Model II ($\hat{\lambda}_1 = 0.862$,$\hat{\lambda}_3 = 0.067$)

		$n = 639$	
		Test statistic value	p-value
Mirrored $T_{PN}^{(SII)}$	$t_1 = -0.9,\ t_2 = -0.9$	165.966	0.054
	$t_1 = -0.5,\ t_2 = -0.5$	-359.286	0.932
	$t_1 = -0.01,\ t_2 = -0.01$	-135.242	0.914
	$t_1 = 0.01,\ t_2 = 0.01$	-133.630	0.899
	$t_1 = 0.5,\ t_2 = 0.5$	-126.924	0.763
	$t_1 = 0.9,\ t_2 = 0.9$	-220.890	0.558
Mirrored $T_{SPN}^{(SII)}$	—	4.237	0.544
$T_{PN}^{(.)}$	$a_1 = -0.9,\ a_2 = -0.9$	1057.191	0.99
	$a_1 = -0.5,\ a_2 = -0.55$	100.903	0.98
	$a_1 = -0.01,\ a_2 = -0.01$	24.906	0.87
	$a_1 = 1,\ a_2 = 1$	3.786	0.91
	$a_1 = 3,\ a_2 = 5$	1.178	0.01
	$a_1 = -0.9,\ a_2 = 5$	152.5798	0.007
Mirrored $T_{PN}^{(SII)}$	$a_1 = -0.9,\ a_2 = -0.9$	78.337	0.40
	$a_1 = 1,\ a_2 = 1$	4.049	0.91
	$a_1 = -0.9,\ a_2 = 5$	1.438	0.20
$FI_n^{(.)}$	—	2.289	0.986
Mirrored $FI_n^{(SII)}$	—	3.443	0.3
Chi-square $^{(.)}$	—	586	$--$

The tests $T_{PN}^{(SII)}$ on neighbourhood of 0, $T_{SPN}^{(SII)}$, $T_{PN}^{(.)}$ (large than -0.9), $FI_n^{()}$ and

$FI_n^{(SII)}$ suggests that the Accidents and Fatalities data fits well the bivariate pseudo-Poisson Full and its mirrored sub-model II, which is inline with the AIC values listed on pages 16 & 18 of Arnold and Manjunath's [2].

5. Conclusion

The GoF tests for the bivariate pseudo-Poisson and its sub-models were the main emphasis of the current note. Based on p.g.f., moments, and Chi-square tests, we proposed a few GoF tests. The test based on the bivariate Fisher index of dispersion-based GoF test is a new contribution to the bivariate count variables. The Supremum of the absolute difference between the calculated p.g.f. and its empirical equivalent is the robust GoF test, i.e., robust to the choice of the alternative distributions. Additionally, we took into account a few existing tests that depend on the estimated p.g.f. and its empirical results, such as K&K, Munoz, and Gamero approaches. Finally, the Chi-square GoF test results for the pseudo-Poisson data were also examined.

A finite sample, a fairly large sample, and asymptotic distributions of test statistics are examined for each of the tests discussed. In addition, we looked at the power and efficacy of each statistical test using the bivariate Conway-Maxwell Poisson and the bivariate Classical Poisson (BCP) as alternative distributions. It has been demonstrated that a test based on the Supremum and Index of dispersion is reliable, consistent, and satisfying. Particularly, the Supremum-based test proved to be more robust to the choice of alternative distributions. Additionally, we suggest utilizing the Munoz and Gamero (M&G) test for moderately small samples and the Supremum (robust) and dispersion tests for moderately large samples. Due to the asymptotic distribution of the test statistic, we also recommend K&K and dispersion tests for sufficiently large data sets. Also, due to its robust property, we suggest considering the Supremum and Chi-square GoF tests if there are no reasonable alternatives to the hypothesis.

The bivariate pseudo-Poisson distribution has been highly advised as the primary choice when modeling bivariate count data whenever the marginals exhibit equal and over-dispersed, see Arnold and Manjunath [2]. The GoF tests that have been suggested will unquestionably add yet another tool for evaluating the compatibility of the bivariate count data. Briefly said, writers are working on an R package that covers fitting (classical and Bayesian analysis) and testing for the bivariate pseudo-Poisson model. The developed package will merit a spot in the toolkit of contemporary modellers because of its simple structure and fast computation.

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6. Appendices

Appendix A. Marginal probability of Y

For the marginal distribution of Y, the probability that $Y=0$ can be computed as

$$P(Y=0)=G_Y(0)=e^{-\lambda_2}e^{\lambda_1(e^{-\lambda_3}-1)}. \tag{29}$$

For the probability that $Y=1$ we have

$$\frac{d}{dt}G_Y(t_2)=G_Y(t_2)\left[\lambda_1\lambda_3 e^{\lambda_3(t_2-1)}+\lambda_2\right]$$

$$P(Y=1)=\frac{\frac{d}{dt}G_Y(t_2)|_{t_2=0}}{1!}=G_Y(0)\left[\lambda_1\lambda_3 e^{-\lambda_3}+\lambda_2\right]. \tag{30}$$

Similarly, $P(Y=2)$ is given as

$$\frac{d^2}{dt^2}G_Y(t_2)=G_Y(t_2)\left[\left(\lambda_1\lambda_2 e^{\lambda_2(t_2-1)}+\lambda_2\right)^2+\lambda_1\lambda_3^2 e^{\lambda_3(t_2-1)}\right]$$

$$P(Y=2)=\frac{\frac{d}{dt}G_Y(t_2)|_{t_2=0}}{2!}=\frac{G_Y(0)}{2!}\left[\left(\lambda_1\lambda_2 e^{-\lambda_2}+\lambda_2\right)^2+\lambda_1\lambda_3^2 e^{-\lambda_3}\right], \tag{31}$$

and finally $P(Y=3)$ is models

$$\begin{aligned}\frac{d^3}{dt^3}G_Y(t_2)= \quad & G_Y(t_2)\Big[\lambda_1\lambda_3(\left(\lambda_1\lambda_3 e^{\lambda_3(t_2-1)}+\lambda_2\right)^2+\lambda_3\\ &(2\left(\lambda_1\lambda_3 e^{\lambda_3(t_2-1)}+\lambda_2\right)+\lambda_3(1+\lambda_1 e^{\lambda_3(t_2-1)})))e^{\lambda_3(t_2-1)}+\\ &\lambda_2(\left(\lambda_1\lambda_3 e^{\lambda_3(t_2-1)}+\lambda_2\right)^2+\lambda_1\lambda_3^2 e^{\lambda_3(t_2-1)})\Big].\end{aligned} \tag{32}$$

$$\begin{aligned}P(Y=3)=\frac{1}{3!}\frac{d^3}{dt^3}G_{X_2}(0)|_{t_2=0}= \quad & \frac{G_Y(0)}{6}\Big[\lambda_1\lambda_3(\left(\lambda_1\lambda_3 e^{-\lambda_3}+\lambda_2\right)^2+\lambda_3\\ &(2\left(\lambda_1\lambda_3 e^{-\lambda_3}+\lambda_2\right)+\lambda_3(1+\lambda_1 e^{-\lambda_3})))e^{-\lambda_3}+\\ &\lambda_2(\left(\lambda_1\lambda_3 e^{-\lambda_3}+\lambda_2\right)^2+\lambda_1\lambda_3^2 e^{-\lambda_3})\Big].\end{aligned} \tag{33}$$

On similar line one can extend the above procedure to get albeit complicated values for the probability that Y assumes any positive value.

Appendix B. Other conditional distribution of the bivariate pseudo-Poisson

In the following we are deriving other conditional distribution, i.e., conditional distribution of X given $Y = y$ by induction for the sub-model II. Consider the joint mass function of pseudo-Poisson sub-model II

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_3 x}(\lambda_3 x)^y}{y!} & x = 1,2,...; y = 0,1,2,... \\ e^{-\lambda_1} & (x,y) = (0,0). \end{cases}$$

Now, consider the case in which $y = 0$ then for each $x = 0,1,2,...$ the conditional mass function will be

$$\begin{aligned} f_{x|Y}(x|0) &= \frac{P(X=x, Y=0)}{P(Y=0)} \\ &= \frac{e^{-\lambda_1 e^{-\lambda_3}}(\lambda_1 e^{-\lambda_3})^x}{x!}. \end{aligned} \tag{34}$$

Indeed the above conditional mass function is a Poisson distribution with mean equal to $\lambda_1 e^{-\lambda_3}$.

Next, consider the case with $y = 1$. For each $x = 1,2,...$ we have

$$\begin{aligned} f_{x|Y}(x|1) &= \frac{P(X=x, Y=1)}{P(Y=1)} \\ &= \frac{e^{-\lambda_1 e^{-\lambda_3}}(\lambda_1 e^{-\lambda_3})^{x-1}}{(x-1)!}, \end{aligned} \tag{35}$$

which is recognizable as the distribution of 1 plus a Poisson($\lambda_1 e^{-\lambda_3}$).

For $y \geq 1$ and for each $x = 1,2,...$ we have a

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{P(X=x, Y=y)}{P(Y=y)} \\ &= \frac{\frac{e^{-\lambda_1 e^{-\lambda_3}}(\lambda_1 e^{-\lambda_3})^{x-1}x^y}{(x-1)!}}{\mu_y}, \end{aligned} \tag{36}$$

where μ_y is the yth moment of a Poisson($\lambda_1 e^{-\lambda_3}$) variable. Note that the expression μ_y can also be expressed in terms of factorial moments and the yth factorial moment is $(\lambda_1 e^{-\lambda_3})^y$. Thus we have

$$\mu_y = \sum_{j=0}^{y} S(y,j)(\lambda_1 e^{-\lambda_3})^y, \tag{37}$$

where $S(y,j)$ is a Stirling number of the second kind. Also note that if $y \geq 1$ then $S(y,0) = 0$.

Appendix C. Examples

Consider the following examples:

Example 6.1. define $w_1(t_1,t_2) = c_1 + c_2 t_1 t_2 + c_3 t_1^2 t_2^2$, $(t_1,t_2)^T \in [0,1]^2$, $c_1, c_2, c_3 \in \mathbb{R}$ and $T_{n,w_1}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$ is

$$
\begin{aligned}
& c_1\Bigg\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+1)(Y_i+Y_j+1)}\right) \\
+ & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Bigg(\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3) \\
& \int_0^1\int_0^1 t_1^{k+m}t_2^{l+n}dt_1dt_2\Bigg) \\
& -2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)\int_0^1\int_0^1 t_1^{x+X_i}t_2^{y+Y_i}dt_1dt_2\right)\Bigg\} + \\
& c_2\Bigg\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+2)(Y_i+Y_j+2)}\right) \\
+ & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Bigg(\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3) \\
& \int_0^1\int_0^1 t_1^{k+m+1}t_2^{l+n+1}dt_1dt_2\Bigg) - \\
& -2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)\int_0^1\int_0^1 t_1^{x+X_i+1}t_2^{y+Y_i+1}dt_1dt_2\right)\Bigg\} + \\
& c_3\Bigg\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+3)(Y_i+Y_j+3)}\right) \\
+ & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Bigg(\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3) \\
& \int_0^1\int_0^1 t_1^{k+m+2}t_2^{l+n+2}dt_1dt_2\Bigg) \\
& -2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)\int_0^1\int_0^1 t_1^{x+X_i+2}t_2^{y+Y_i+2}dt_1dt_2\right)\Bigg\}, \quad (38)
\end{aligned}
$$

where $\mathscr{P}(i;.\hat{\lambda})$ is a Poisson probability at i for the estimated parameter $\hat{\lambda}$. Further simplification gives us

$$
\begin{aligned}
T_{n,w_1}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3) \quad = \quad & \frac{c_1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+1)(Y_i+Y_j+1)}\right)+ \\
& \frac{c_2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+2)(Y_i+Y_j+2)}\right)+ \\
& \frac{c_3}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+3)(Y_i+Y_j+3)}\right)+ \\
& c_1\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3)}{(k+m+1)(l+n+1)}\right)+ \\
& c_2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3)}{(k+m+2)(l+n+2)}\right)+ \\
& c_3\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3)}{(k+m+3)(l+n+3)}\right)+ \\
& -2c_1\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\frac{\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)}{(x+X_i+1)(y+Y_i+1)}\right) \\
& -2c_2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\frac{\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)}{(x+X_i+2)(y+Y_i+2)}\right) \\
& -2c_3\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\frac{\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)}{(x+X_i+3)(y+Y_i+3)}\right). \qquad (39)
\end{aligned}
$$

We refer to Table 1 and Figure 2 for the quantile values and frequency distribution for $a_1 = 1$ and $a_2 = 1$, respectively.

Example 6.2. For a general form of $w(.,.)$, consider $w_2(t_1,t_2) = t_1^{a_1}t_2^{a_2}$, $(t_1,t_2)^T \in [0,1]^2$, $a_1, a_2 \in (-1,\infty)$, which allows us to include a negative powers as well, then the T_{n,w_2} is

$$
\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+a_1+1)(Y_i+Y_j+a_2+1)}\right) \\
+ \quad & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Bigg(\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n;\hat{\lambda}_2+m\hat{\lambda}_3) \\
& \int_0^1\int_0^1 t_1^{k+m+a_1}t_2^{l+n+a_2}dt_1dt_2\Bigg) \\
& -2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)\int_0^1\int_0^1 t_1^{x+X_i+a_1}t_2^{y+Y_i+a_2}dt_1dt_2\right).
\end{aligned}
$$

Now, further simplification will give us closed form expression for the statistic

$$\begin{aligned} T_{n,w_2}(\hat{\lambda}_1, \hat{\lambda}_2\hat{\lambda}_3) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{(X_i+X_j+a_1+1)(Y_i+Y_j+a_2+1)}\right) \\ &+ \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\mathscr{P}(k;\hat{\lambda}_1)\mathscr{P}(l;\hat{\lambda}_2+k\hat{\lambda}_3)\mathscr{P}(m;\hat{\lambda}_1)\mathscr{P}(n\mathscr{P}(x;\hat{\lambda}_1)}{(k+m+a_1+1)(l+n+a_2+1)}\right) \\ &\quad -2\sum_{i=1}^{n}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left(\frac{\mathscr{P}(x;\hat{\lambda}_1)\mathscr{P}(y;\hat{\lambda}_2+x\hat{\lambda}_3)}{(x+X_i+a_1+1)(y+Y_i+a_2+1)}\right). \qquad (40) \end{aligned}$$

We refer to Table 2 and Figure 3 for the quantile values and frequency distribution for $a_1 = 1$ and $a_2 = 1$, respectively. Also, with varying a_1 and a_2 values finite sample distribution of the statistics, see Table 3 and Figures 4, 5 and 6.